Lecture 8: 1D and 2D Discrete transforms and introduction to wavelets

Learning Objectives:

- Performing 2D discrete Fourier transforms in Matlab
- Generalized basis functions, hybrid space and “basis” images
- Introduction to wavelet transforms

Assignment:

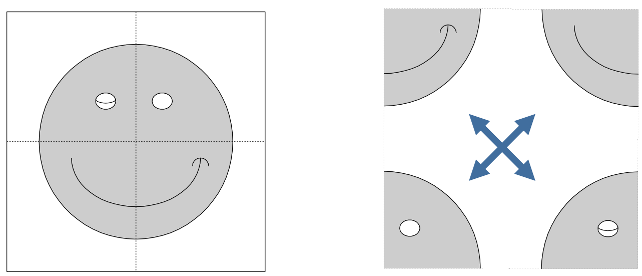
1. Read Chapter 7 of Digital Image Processing Using MATLAB, titled “Wavelets”

I. 2D DFT shift example in Matlab (courtesy Dr. Wieben):

fftshift in 2D

Use fftshift for 2D functions

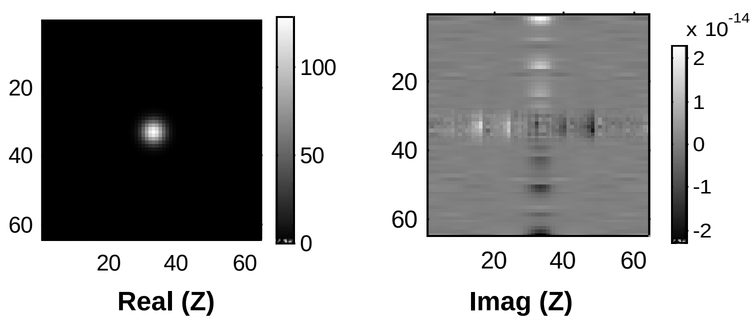
- **>>smiley2 = fftshift(smiley);**



2D Discrete Fourier Transform

- **>>Z=fftshift(fft2(fftshift(z)));**
- **>>whos**
- **>>figure**
- **>>imshow(real(Z))**
- **>>imshow(real(Z),[])**
- **>>colorbar**

- **>>figure**
- **>>imshow(imag(Z),[])**
- **>>colorbar**

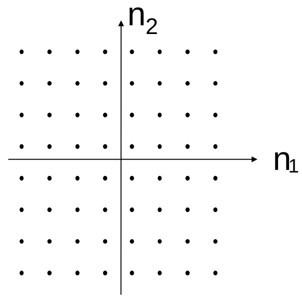


Lecture 8: 1D and 2D Discrete transforms and introduction to wavelets

2D Sampling/Discrete-Space Signal

$x(n_1, n_2)$ *where* $\frac{-N_1}{2} \le n_1 \le \frac{N_1}{2} - 1$

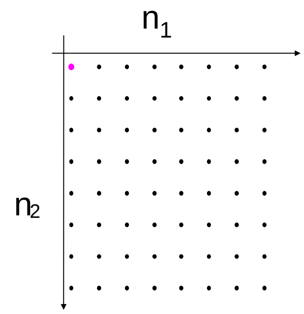
$\frac{-N_2}{2} \le n_2 \le \frac{N_2}{2} - 1$



2D Sampling/Discrete-Space Signal

$f(n_1, n_2)$ *where* $1 \le n_1 \le N_1$

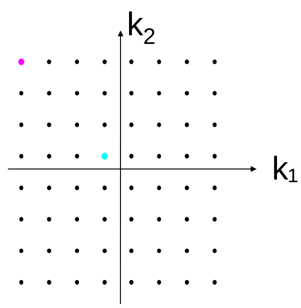
$1 \le n_2 \le N_2$



2D Discrete Fourier Transform

$F(k_1, k_2)$ *where* $1 \le k_1 \le N_1 - 1$

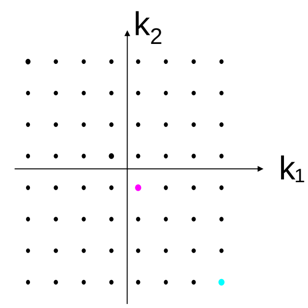
$1 \le k_2 \le N_2 - 1$



2D Discrete Fourier Transform

$F(k_1, k_2)$ *where* $\frac{-N_1}{2} \le k_1 \le \frac{N_1}{2} - 1$

$\frac{-N_2}{2} \le k_2 \le \frac{N_2}{2} - 1$



Lecture 8: 1D and 2D Discrete transforms and introduction to wavelets
DCT case more explicitly:

Recall 2D DCT:

$$\mathrm{N}\times\mathrm{N}\ \text{point}\ f(n_1,n_2)\leftrightarrow 2\mathrm{N}\times 2\mathrm{N}\ \text{point}\ g(n_1,n_2)\Leftrightarrow 2\mathrm{N}\times 2\mathrm{N}\ \text{point}\ G(k)\leftrightarrow \mathrm{N}\times\mathrm{N}\ \text{point}\ C(k)$$

$$f(n_1,n_2)=\frac{1}{\sqrt{N_1N_2}}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\alpha_1(k_1)\alpha_2(k_2)C(k_1,k_2)\cos\left[\frac{\pi}{2N_1}k_1(2n_1+1)\right]\cos\left[\frac{\pi}{2N_2}k_2(2n_2+1)\right],\qquad 0\le n_1\le N_1-1,$$

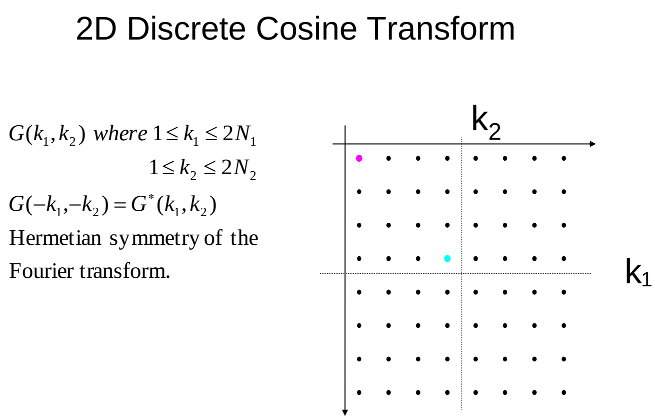
$0\le n_2\le N_2-1,$

$$\textit{and } 0 \textit{ otherwise}.\qquad \alpha_{1,2}(k_{1,2})=\begin{cases}1/2 & k_{1,2}=0\\ 1 & 1\le k_{1,2}\le N-1\end{cases}$$

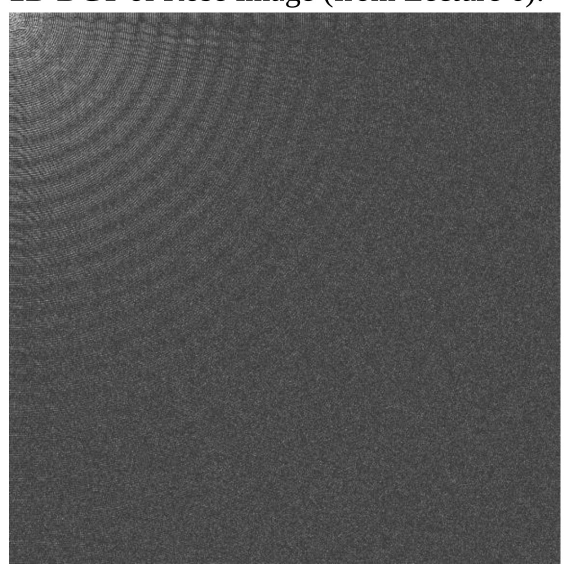
$$C(k_1,k_2)=\frac{1}{\sqrt{N_1N_2}}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}4f(n_1,n_2)\cos\left[\frac{\pi}{2N_1}k_1(2n_1+1)\right]\cos\left[\frac{\pi}{2N_2}k_2(2n_2+1)\right],\ 0\le k_1\le N_1-1,$$

$0\le k_2\le N_1-1$

and 0 *otherwise*.



2D DCT of Rose Image (from Lecture 6):



Lecture 8: 1D and 2D Discrete transforms and introduction to wavelets

II. Generalized 1D and 2D transforms:

General Requirements of a Basis Set:

1. Review of linear transforms
- Definition of a transform kernel
  - Defines a finite set of vectors
  - Span an n-dimensional vector space
    - The kernel is a set of vectors
  - Linearly independent
    - Any vector $\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + \ldots$
  - Often impose orthogonality
    - Inner product $< \mathbf{v}_1, \mathbf{v}_2 > = 0$
      - $<\mathbf{u},\mathbf{v}> = \mathbf{u}_1 \mathbf{v}_1 + \mathbf{u}_2 \mathbf{v}_2 + \mathbf{u}_3 \mathbf{v}_3 + \ldots$
  - And orthonormality
    - Inner product $< \mathbf{v}_1, \mathbf{v}_1 > = 1$.

2. Subset includes linear integral transforms:

$$F(\alpha) = \int_a^b T(\alpha, t) f(t) dt = < f, T(\alpha t) > \equiv \text{"Inner Product"},$$

where $F(\alpha)$ is the transform of f(t) with respect to the kernel $T(\alpha, t)$, and α is the tranform variable.

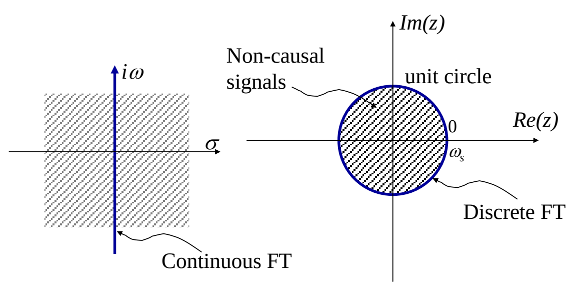
Examples:

- *$T(i\omega,t) = e^{-i\omega t}$, Fourier Transform*
- *$T(k,x) = x J_n(kx)$, Hankel Transform*
  where $J_n(kx)$ are the n^{th} order Bessel functions
- *$T(s,t) = e^{-st}$, Laplace Transform*
  where the s can in general be complex valued:

Laplace to z-Transform

$$F(s) = \int_a^b e^{-st} f(t) dt, \qquad H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$s = \sigma + i\omega$$



2. Discrete forms can be represented as a matrix/array computation:

x is an N by 1 vector and **T** is an N by N matrix, then

$$y_i = \sum_{j=0}^{N-1} t_{i,j} x_j \quad \text{or } \mathbf{y} = \mathbf{Tx}$$

where there is a matrix inverse T^{-1} such that

$$\mathbf{x} = \mathbf{T}^{-1}\mathbf{y}$$

where T is nonsingular by virtue of forming a linearly independent basis set that "spans" the vector space.

Kernel T is Orthonormal:

If **T** is a unitary matrix, then

$$\mathbf{T}^{-1} = \mathbf{T}^{*t} \text{ and } \mathbf{T}\mathbf{T}^{*t} = \mathbf{I}$$

where the rows of the kernel matrix T form a set of basis vectors for an N-dimensional vector space:

$$\sum_{i=0}^{N-1} T_{j,i} T^*_{k,i} = \delta_{j,k}.$$

where “*” is the complex conjugate and “t” is the transpose.

Examples:

1DFT:

f is an N by 1 vector and W is an N by N matrix, then

$$w_{n,k} = \frac{1}{\sqrt{N}} e^{-i2\pi\frac{nk}{N}}$$

$$W = \begin{bmatrix} w_{0,0} & \cdots & w_{0,N-1} \\ \vdots & \ddots & \\ w_{N-1,0} & \cdots & w_{N-1,N-1} \end{bmatrix}$$

where

$$\mathbf{F} = W\boldsymbol{f} \text{ and } \boldsymbol{f} = W^{*t}\boldsymbol{F}$$

where **f** and **F** are N by 1 signal and spectrum vectors.

2DFT:

F is an N by N matrix and W is the same N by N matrix, where

$$w_{n,k} = \frac{1}{\sqrt{N}} e^{-i2\pi\frac{nk}{N}}$$

$$W = \begin{bmatrix} w_{0,0} & \cdots & w_{0,N-1} \\ \vdots & \ddots & \\ w_{N-1,0} & \cdots & w_{N-1,N-1} \end{bmatrix}$$

only now,

$\mathbf{G} = W\boldsymbol{F}W$ and $\boldsymbol{F} = W^{*t}\boldsymbol{G}W^{*t}$

where **F** is the image matrix and **G** is the spectrum matrix.

Separability and "hybrid space":

if $\Im(i,k,m,n)$ is separable then

$\Im(i,k,m,n) = T_r(i,m)T_c(k,n)$

carried out in a row-wise operation followed by

a column-wise operation.

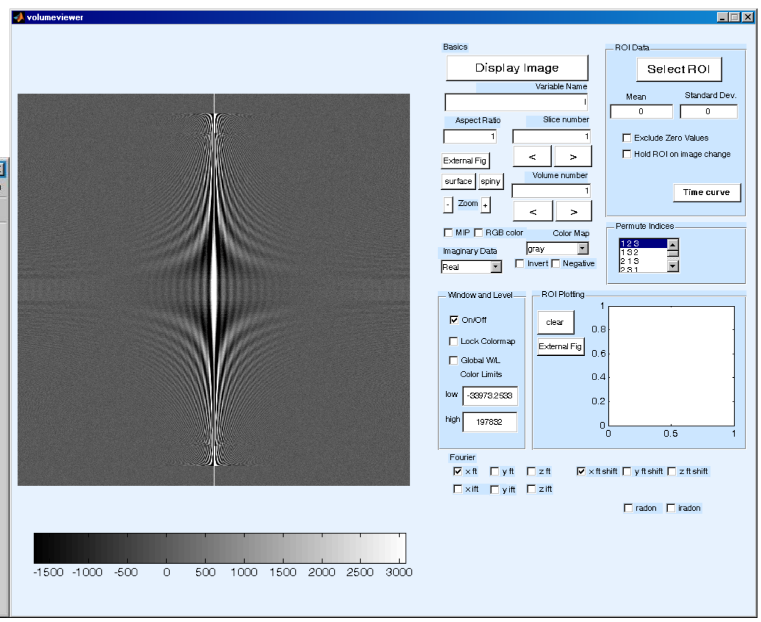
For $\Im(i,k,m,n) = T(i,m)T(k,n)$

$$G_{m,n} = \sum_{i=0}^{N-1} T(i,m)\left[\sum_{k=0}^{N-1} F_{i,k}T(k,n)\right] = \mathbf{G} = \mathbf{TFT}$$

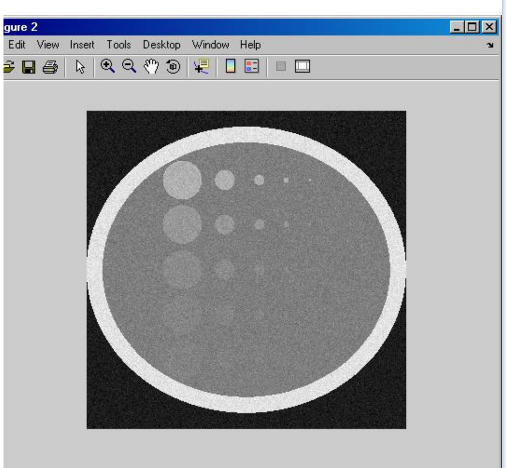
and the inverse transform is

$$\mathbf{F} = \mathbf{T}^{-1}\mathbf{G}\mathbf{T}^{-1} = \mathbf{T}^{*t}\mathbf{G}\mathbf{T}^{*t}$$

Hybrid DFT space (DFT row-wise):



Rose image Example:



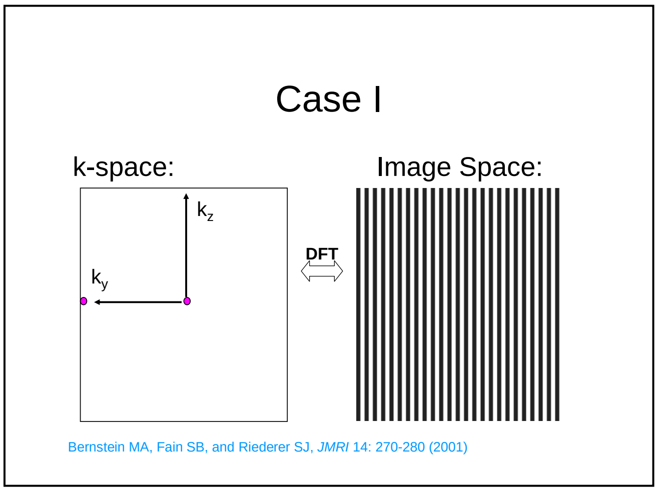
$$W = \begin{bmatrix} w_{0,0} & \cdots & w_{0,N-1} \\ \vdots & \ddots & \\ w_{N-1,0} & \cdots & w_{N-1,N-1} \end{bmatrix} \cdot \begin{bmatrix} w^*_{0,0} & \cdots & w^*_{N-1,0} \\ \vdots & \ddots & \\ w^*_{0,N-1} & \cdots & w^*_{N-1,N-1} \end{bmatrix}$$

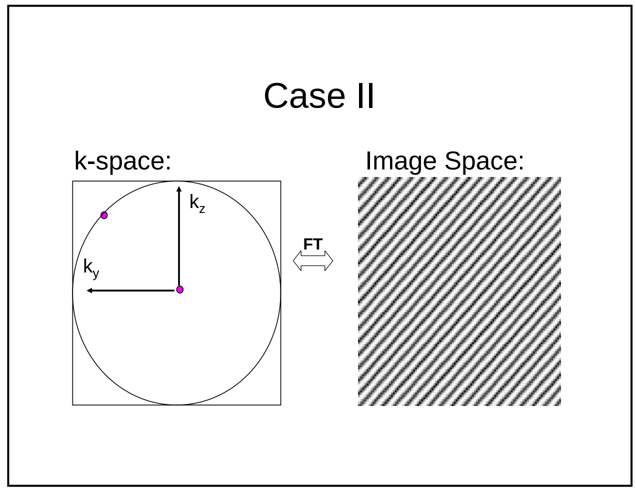
only now,

$$\mathbf{B} = WW^{*t}$$

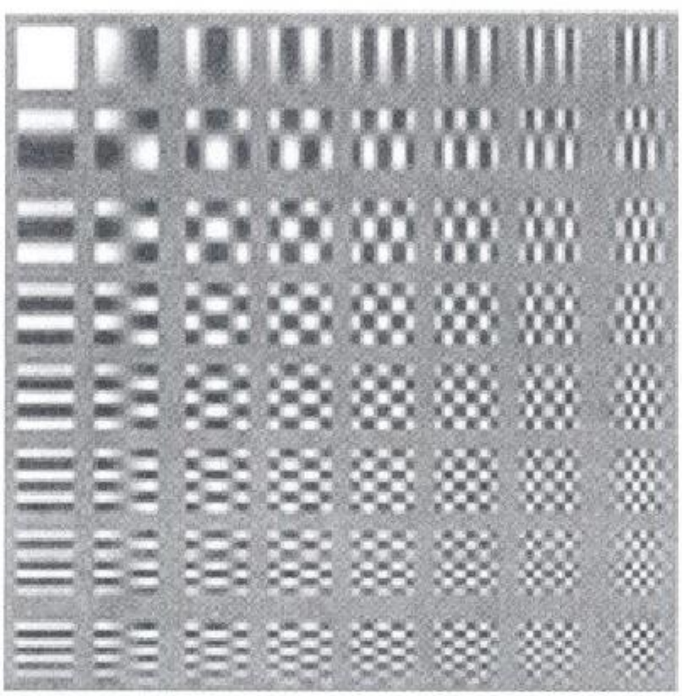
is a set of basis 2D arrays with $\delta(n_1, n_2)$ and 0 otherwise.

In conjugate image space, these form the "basis images" of the 2D DFT.





Matrix of basis images of the Discrete Cosine Transform:



(From Wandell)

Lecture 8: 1D and 2D Discrete transforms and introduction to wavelets

Example: The Gabor transform:
Daugman, 2000, University of Cambridge, The Computer Laboratory, Cambridge CB2 3QG, U.K., www.CL.cam.ac.uk/users/jgd1000/

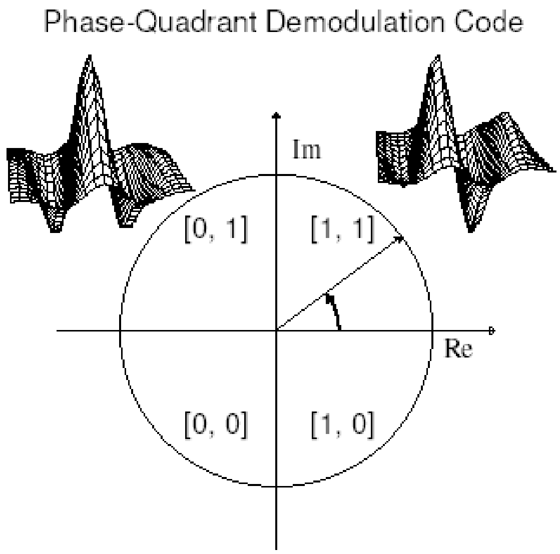
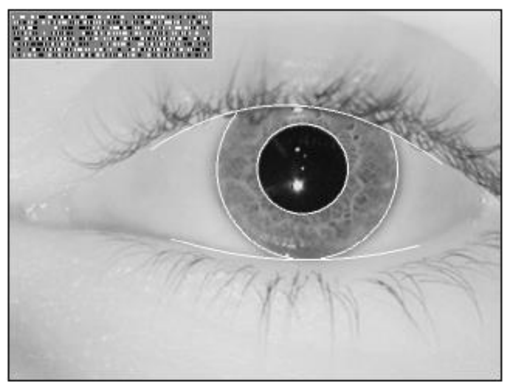


Figure 2: The phase demodulation process used to encode iris patterns. Local regions of an iris are projected (Eqt 2) onto quadrature 2D Gabor wavelets, generating complex-valued coefficients whose real and imaginary parts specify the coordinates of a phasor in the complex plane. The angle of each phasor is quantized to one of the four quadrants, setting two bits of phase information. This process is repeated all across the iris with many wavelet sizes, frequencies, and orientations, to extract 2,048 bits.



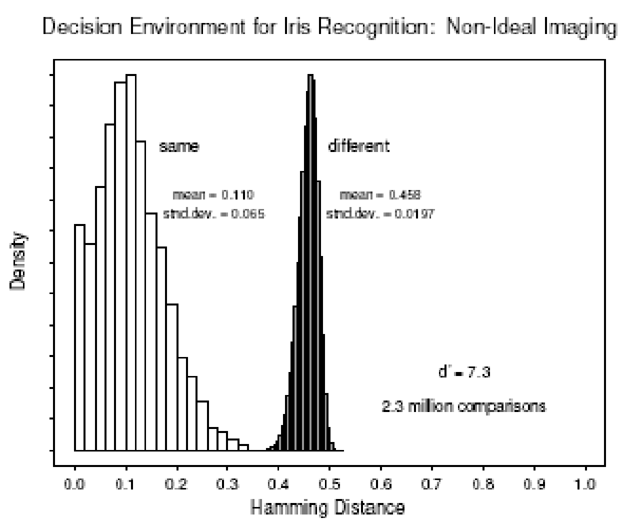


Figure 9: The Decision Environment for iris recognition under relatively unfavourable conditions, using images acquired at different distances, and by different optical platforms.

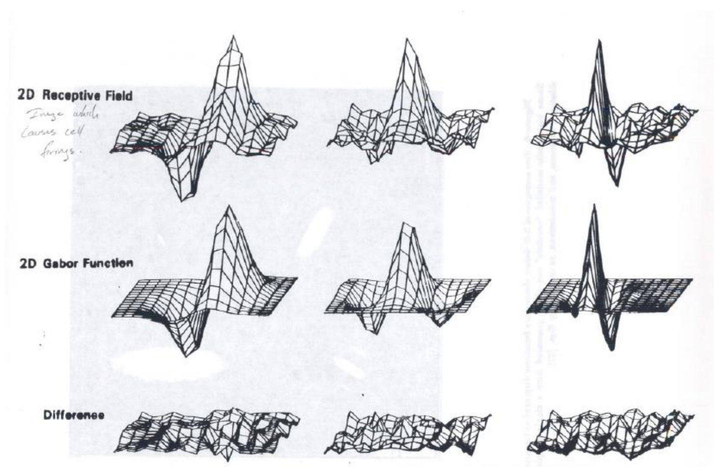
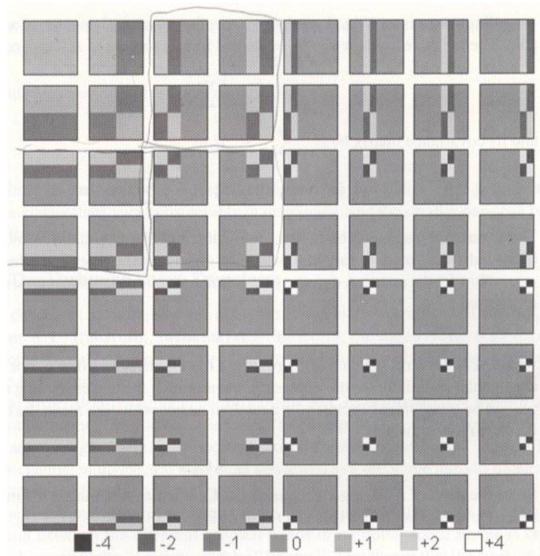


Figure 5. Top row: illustrations of empirical 2-D receptive field profiles measured by J.P. Jones and L.A. Palmer (personal communication) in simple cells of the cat visual cortex. Middle row: best-fitting 2-D Gabor elementary function for each neuron, described by Eqt. [10]. Bottom row: residual error of the fit, indistinguishable from random error in the Chi-squared sense for 97% of the cells studied.

Lecture 8: 1D and 2D Discrete transforms and introduction to wavelets
Haar Wavelets

- Symmetric separable unitary transform
  - Haar functions form the basis
- Haar functions vary in both scale (width) and position
  - Whereas FT basis functions vary only in frequency



$$h_0(x) = \frac{1}{\sqrt{N}}$$

$$h_k(x) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & \frac{q-1}{2^p} \le x < \frac{q-1/2}{2^p} \\ -2^{p/2} & \frac{q-1/2}{2^p} \le x < \frac{q}{2^p} \\ 0 & otherwise \end{cases}$$

Let $x = j/N$ for $j = 0,1,..., N-1$ then we get a set of odd rectangular pulse pairs, except for $k = 0$. Defined on $[0,1]$ with the integer $0 \le k \le N-1$ specified by $k = 2^p + q - 1$.

Hr =

0.3536	0.3536	0.3536	0.3536	0.3536	0.3536	0.3536	0.3536
0.3536	0.3536	0.3536	0.3536	-0.3536	-0.3536	-0.3536	-0.3536
0.5000	0.5000	-0.5000	-0.5000	0	0	0	0
0	0	0	0	0.5000	0.5000	-0.5000	-0.5000
0.7071	-0.7071	0	0	0	0	0	0
0	0	0.7071	-0.7071	0	0	0	0
0	0	0	0	0.7071	-0.7071	0	0
0	0	0	0	0	0	0.7071	-0.7071

First step of the 1D Wavelet transform is the Haar Transform: $d = H_r s$