2D Fourier Transform

Review of 2D Fourier Theory

$$\mathrm{F}(u,v) = \int_{-\infty}^{\infty}\!\!\int \mathrm{f}(x,y)\cdot e^{-i\cdot 2\pi\cdot(ux+vy)}\,dxdy$$

$$\mathrm{f}(x,y) = \int_{-\infty}^{\infty}\!\!\int \mathrm{F}(u,v)\cdot e^{+i\cdot 2\pi\cdot(ux+vy)}\,dudv$$

We view f(*x*,*y*) as a linear combination of complex exponentials that represent plane waves.
F(*u*,*v*) describes the weighting of each wave.

The wave $e^{+i\cdot 2\pi\cdot(ux+vy)}$ has a frequency $\sqrt{u^2+v^2}$

and a direction $\theta = \tan^{-1}\left(\frac{v}{u}\right)$

Review of 2D Fourier theory, continued.

F(u,v) can be plotted as real and imaginary images, or as magnitude and phase. |F(u,v)| = $\sqrt{(\mathrm{Re}(F(u,v))^2+(\mathrm{Im}(F(u,v))^2}$

And Phase(F(u,v) = arctan(Im(F(u,v)/Re(F(u,v))

Just as in the 1D case, pairs of exponentials make cosines or sines.

We can view F(*u*,*v*) as

$$\mathrm{F}(u,v)=\iint_{-\infty}^{\infty}\mathrm{f}(x,y)\cdot\cos(2\pi\cdot(ux+vy))dxdy$$

$$-\mathrm{i}\iint_{-\infty}^{\infty}\mathrm{f}(x,y)\cdot\sin(2\pi\cdot(ux+vy))dxdy$$

Review of 2D Fourier theory, continued: properties

- Similar to 1D properties

Let

$$f(x,y) \leftrightarrow F(u,v)$$

$$g(x,y) \leftrightarrow G(u,v)$$

Then

1. Linearity

$$af + bg \leftrightarrow aF + bG$$

2. Scaling or Magnification

$$g(ax, by) \leftrightarrow \frac{1}{|ab|} G\left(\frac{u}{a}, \frac{v}{b}\right)$$

3. Shift

$$g(x-a, y-b) \leftrightarrow G(u,v) \cdot e^{-i \cdot 2\pi \cdot (au+bv)}$$

4. Convolution

also let

$$h(x,y) \leftrightarrow H(u,v)$$

then,

$$\int_{-\infty}^{\infty}\int g(\varepsilon,\eta) \cdot h(x-\varepsilon, y-\eta) d\varepsilon d\eta <=> G(u,v)H(u,v)$$

Review of 2D Fourier theory, continued: properties

If g(*x*,*y*) can be expressed as $g_x(x)g_y(y)$, the F{g(x,y)} =

$$\int_{-\infty}^{\infty}\int g(x',y')\cdot e^{-i\cdot 2\pi\cdot(ux'+vy')}dx'\,dy'$$

$$= \int_{-\infty}^{\infty} g_x(x')\cdot e^{-i\cdot 2\pi\cdot ux'}dx' \int_{-\infty}^{\infty} g_y(y')\cdot e^{-i\cdot 2\pi\cdot vy'}dy'$$

$$= G_x(u)G_y(v)$$

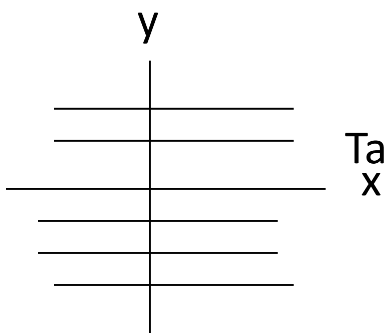
$$\Pi(x,y) = \Pi(x)\Pi(y) \leftrightarrow \mathrm{sinc}(u)\mathrm{sinc}(v)$$

$$\mathrm{comb}(x)\mathrm{comb}(y) \leftrightarrow \mathrm{comb(u)comb(v)}$$

Relation between 1-D and 2-D Fourier Transforms

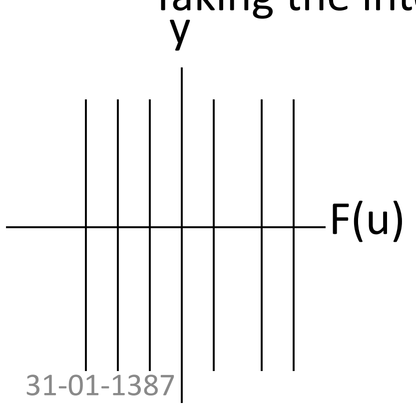
$$F(u,v)=\int_{-\infty}^{\infty} e^{-i\cdot 2\pi\cdot vy}\left[\int_{-\infty}^{\infty} f(x,y)\cdot e^{-i\cdot 2\pi\cdot ux}\,dx\right]dy$$

Rearranging the Fourier Integral,



Taking the integrals along x gives, $\hat{F}(u,y)$

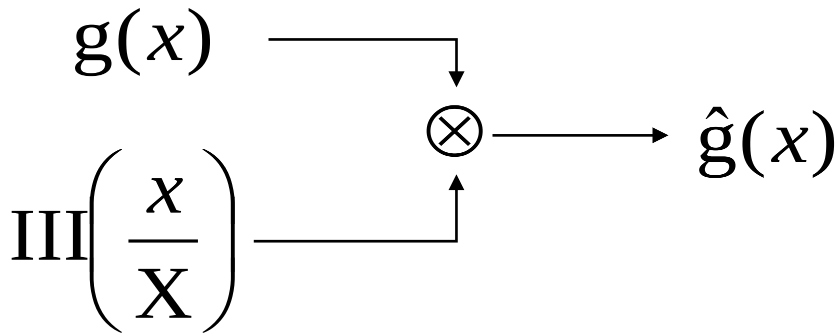
Taking the integrals of $\hat{F}(u,y)$ along y gives F(u,v)



$$F(u,v)=\int_{-\infty}^{\infty}\hat{F}(u,y)e^{-i\cdot 2\pi\cdot vy}\,dy$$

Sampling

Model



We use the comb function and scale it for the sampling interval X.

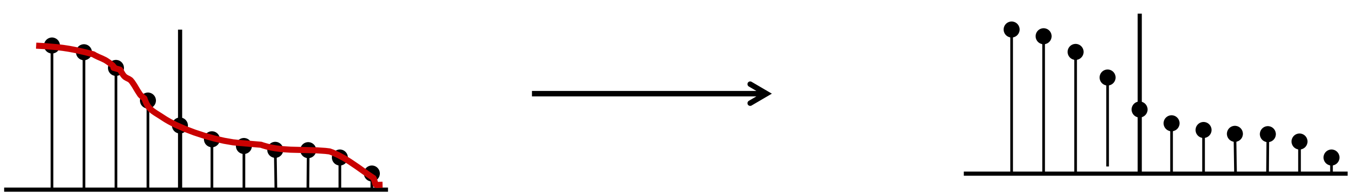
$$\mathrm{III}\left(\frac{x}{X}\right) = \sum \delta\left(\frac{x}{X} - n\right)$$

$$= \sum \delta\left(\frac{1}{X}(x - nX)\right)$$

$$= X \sum \delta(x - nX)$$

$$\hat{g}(x) = g(x) \cdot \mathrm{III}\left(\frac{x}{X}\right)$$

$$= X \sum g(nX)\,\delta(x - nX)$$



Spectrum of Sampled Signal

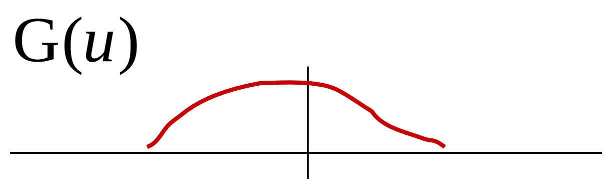
$$\hat{\mathrm{G}}(u) = \mathcal{F}\{\hat{\mathrm{g}}(x)\} = \mathcal{F}\left\{ \mathrm{III}\left(\frac{\mathrm{x}}{\mathrm{X}}\right) \cdot \mathrm{g}(x) \right\}$$

Multiplication in one domain becomes convolution in the other,

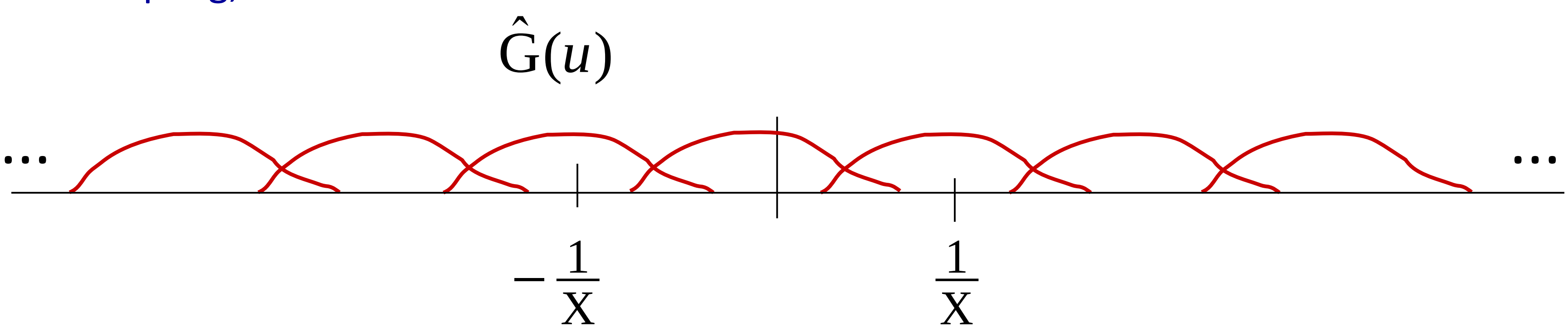
$$\mathrm{X} \cdot \mathrm{III}(\mathrm{X}u) * \mathrm{G}(u) = X \sum \delta\left(Xu - n\right) * \mathrm{G}(u)$$

$$= \mathrm{X} \sum \delta\left(X\left(u - \frac{n}{X}\right)\right) * \mathrm{G}(u) = \sum_{n=-\infty}^{n=\infty} G\left(u - \frac{n}{X}\right)$$

Prior to sampling,



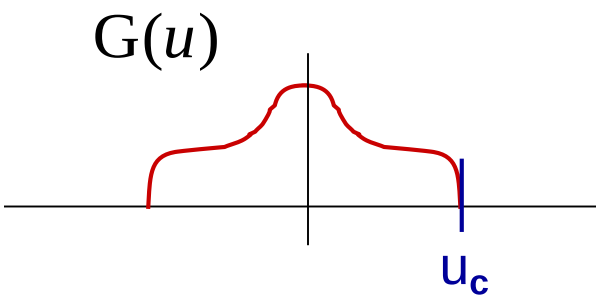
After sampling,



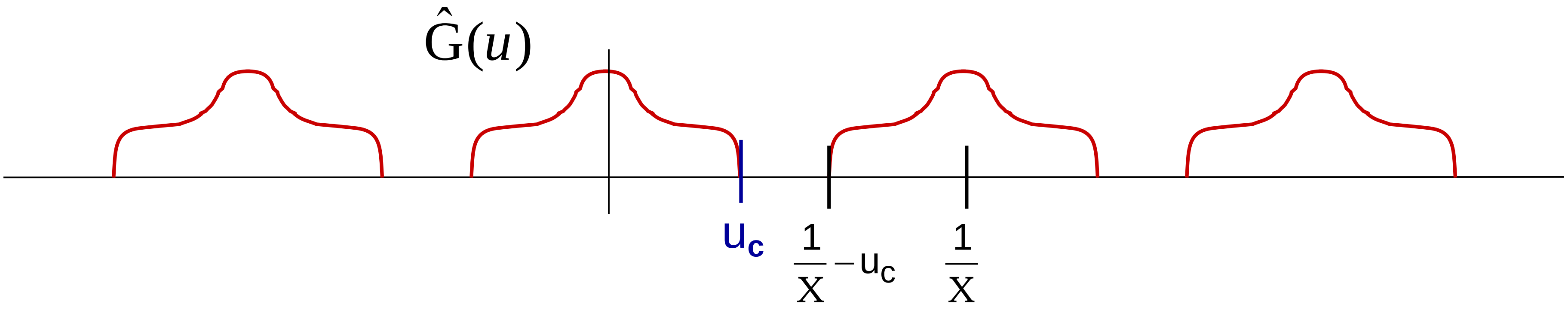
Spectrum of Sampled Signal: Band limiting and Aliasing

If G(u) is band limited to u_c, (cutoff frequency)

G(u) = 0 for $|u| > u_c$.



To avoid overlap (aliasing),

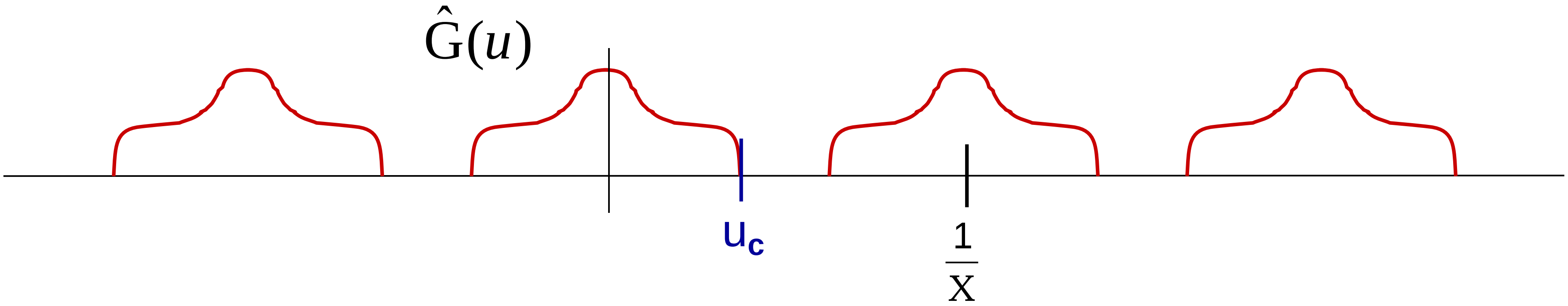


$$\frac{1}{X} - u_c > u_c \qquad \frac{1}{X} > 2 \cdot u_c$$

Nyquist Condition:

Sampling rate must be greater than twice the highest frequency component.

Spectrum of Sampled Signal: Restoration of Original Signal



Can we restore g(x) from the sampled frequency-domain signal?

Yes, using the Interpolation Filter

$$H(u) = \Pi\left(\frac{u}{2u_c}\right)$$

$$\begin{aligned} g(x) &= \hat{g}(x) * h(x) \\ &= \hat{g}(x) * 2u_c \cdot \mathrm{sinc}(2u_c x) \\ &= X \sum_{n=-\infty}^{\infty} g(nX) \cdot \delta(x - nX) * 2u_c \cdot \mathrm{sinc}(2u_c x) \\ &= \sum_{n=-\infty}^{\infty} 2X \cdot u_c \cdot g(nX) \cdot \mathrm{sinc}(2u_c(x - nX)) \end{aligned}$$

Spectrum of Sampled Signal: Restoration of Original Signal(2)

From previous page,

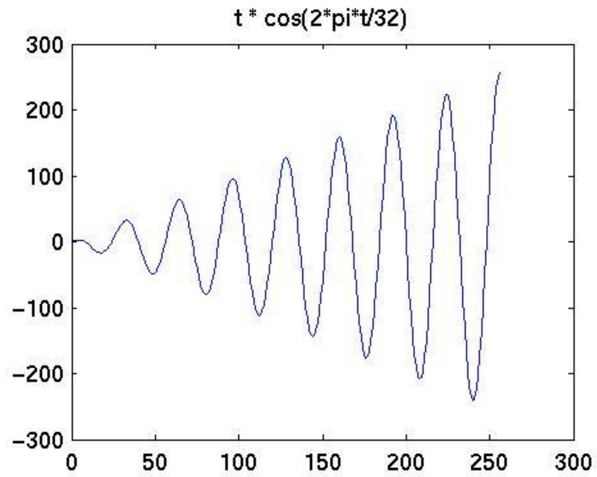
$$g(x) = \sum_{n=-\infty}^{\infty} 2\mathrm{X} \cdot u_{\mathrm{c}} \cdot \mathrm{g}(n\mathrm{X}) \cdot \mathrm{sinc}(2u_{\mathrm{c}}(x - n\mathrm{X}))$$

g(x) is restored from a combination of sinc functions.

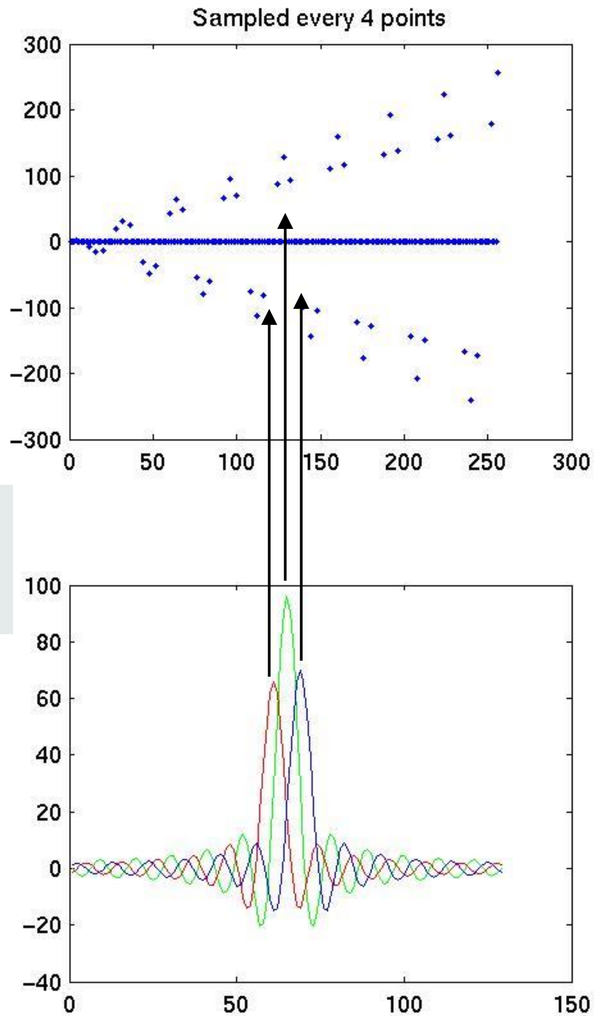
Each is weighted and shifted according to its corresponding sampling point.

Visualizing Sinc Interpolation

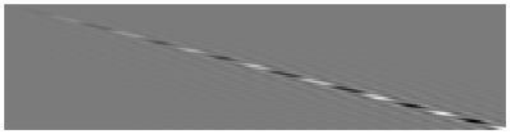
Original function



Sampled function



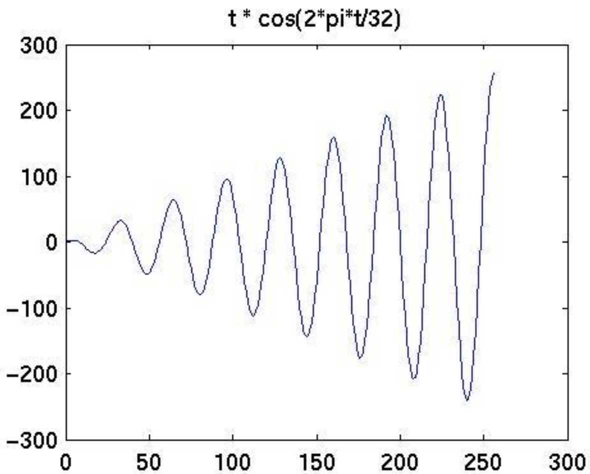
$$g(t) = \sum_{n=-\infty}^{\infty} 2\mathrm{X} \cdot u_{\mathrm{c}} \cdot \mathrm{g}(n\mathrm{X}) \cdot \mathrm{sinc}(2u_{\mathrm{c}}(t - n\mathrm{X}))$$



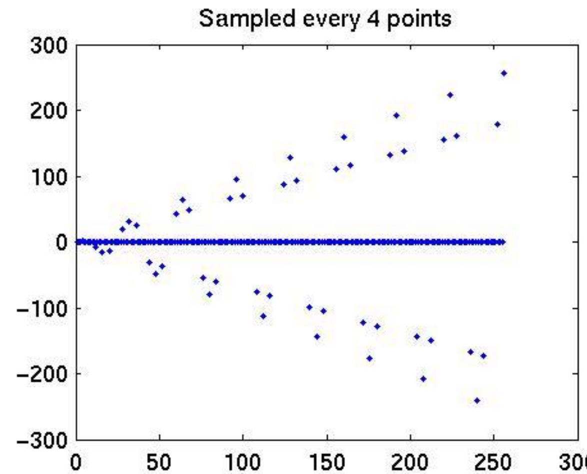
Each row shows convolution of shifted sinc with a sampled point. Sum lines along vertical direction to get output.

Weighted and shifted sincs for 3 sample points shown by black arrows

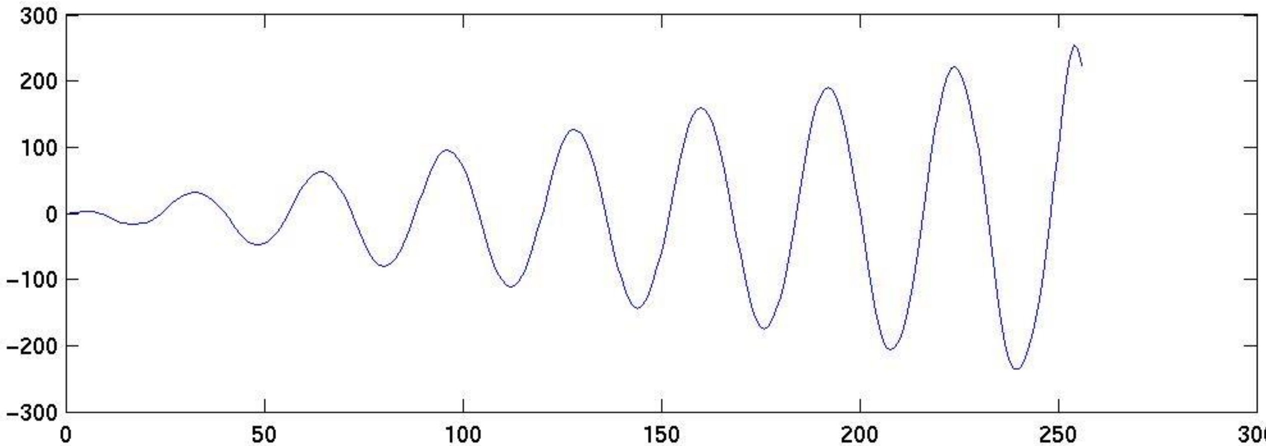
Original functions and output



a)



b)



a) Continuous waveform b) Sampled waveform c) Sinc interpolation of sampled waveform (sum of vertical lines in lower left plot from previous slide.

Two Dimensional Sampling

$$\hat{g}(x,y) = \mathrm{III}\left(\frac{x}{X}\right)\mathrm{III}\left(\frac{y}{Y}\right)g(x,y)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta(x - nX, y - mY)\cdot g(x,y)$$