2D Sampling

Goal: Represent a 2D function by a finite set of points.

- particularly useful to analysis w/ computer operations.

Points are sampled every X in *x*, every Y in *y*.

How will the sampled function appear in the spatial frequency domain?

Two Dimensional Sampling: Sampled function in freq. domain

How will the sampled function appear in the spatial frequency domain?

$$\hat{G}(u,v) = \mathcal{F}\{\hat{g}(x,y)\}$$
$$= XY \cdot \text{III}(uX) \cdot \text{III}(vY) ** G(u,v)$$

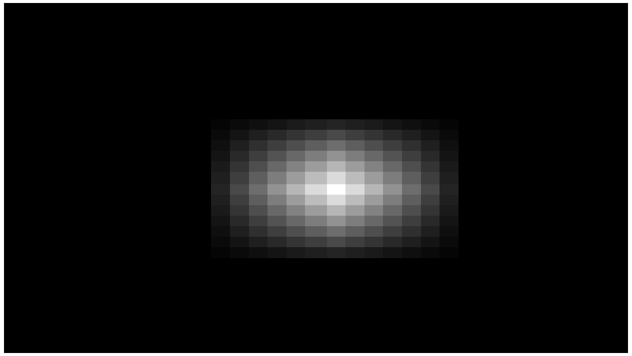
Since

$$XY \cdot \text{comb}(uX) \cdot \text{comb}(vY) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta\left(u - \frac{n}{X}, v - \frac{m}{Y}\right)$$

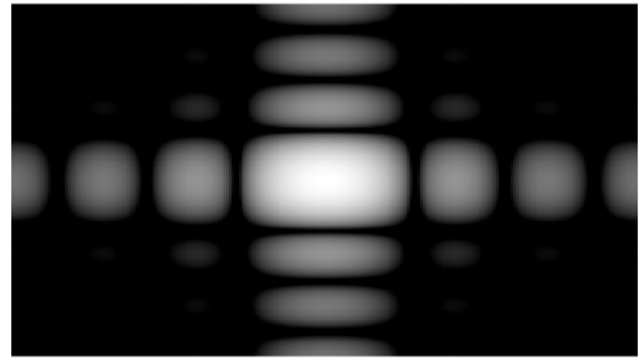
$$\hat{G}(u,v) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G\left(u - \frac{n}{X}, v - \frac{m}{Y}\right)$$

The result: Replicated G(u,v), or "islands" every 1/X in u, and 1/Y in v.

Example



Let g(x,y) =Λ(x/16)Λ(y/16) be a continuous function



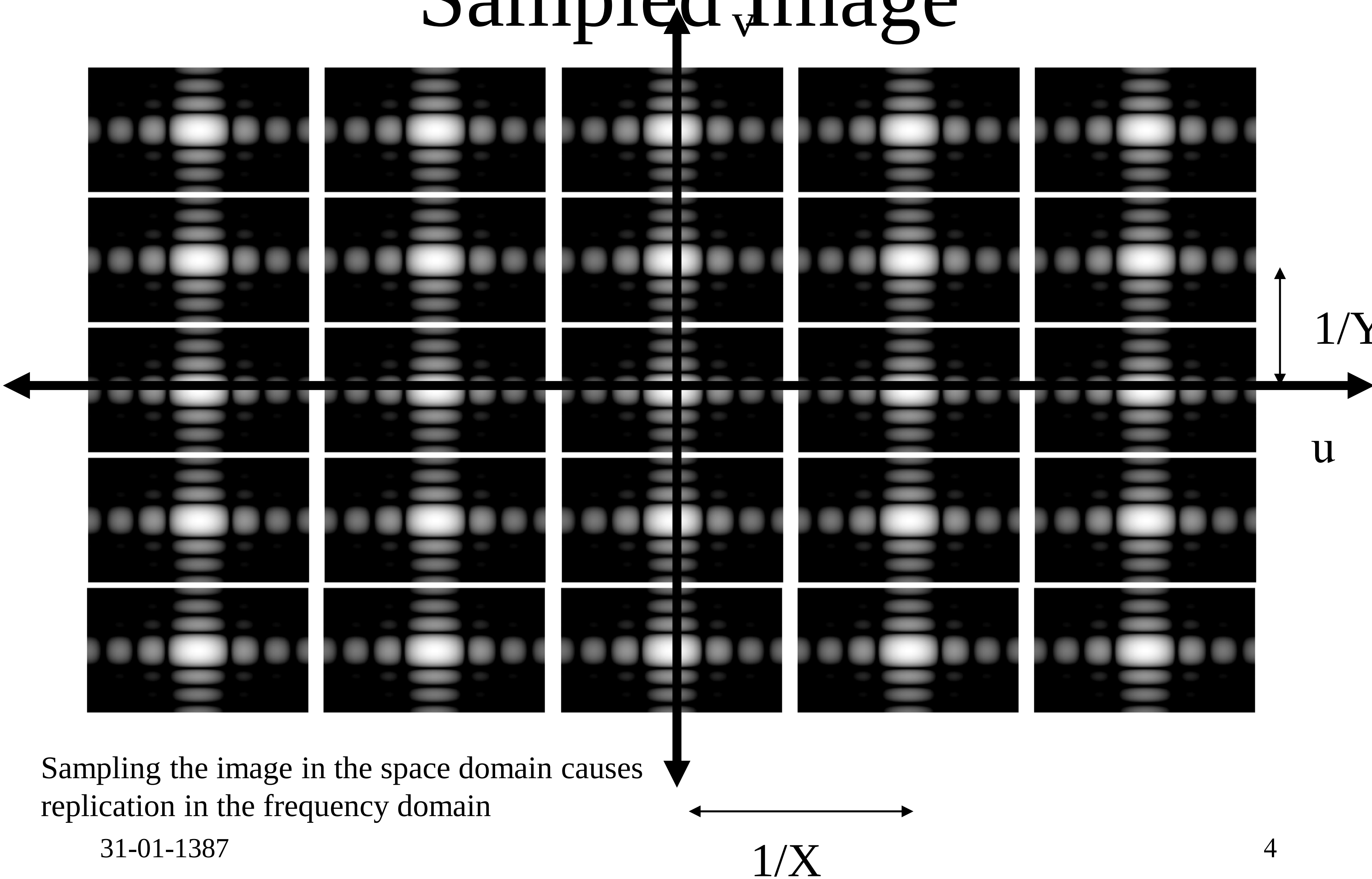
Here we show its continuous transform G(u,v)

Now sampling the function gives the following in the space domain

$$\hat{g}(x, y) = \mathrm{III}\left(\frac{x}{X}\right)\mathrm{III}\left(\frac{y}{Y}\right) g(x, y)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x - nX, y - mY) \cdot g(x, y)$$

Fourier Representation of a Sampled Image



Sampling the image in the space domain causes replication in the frequency domain

Two Dimensional Sampling: Restoration of original function

$$\mathrm{H}(u,v) = \prod(u\mathrm{X}) \cdot \prod(v\mathrm{Y})$$ will filter out unwanted islands.

Let's consider this in the image domain.

$$\hat{\mathrm{g}}(x,y) ** \mathrm{h}(x,y)$$

$$= \left[\mathrm{III}\left(\frac{x}{\mathrm{X}}\right)\mathrm{III}\left(\frac{y}{\mathrm{Y}}\right)\mathrm{g}(x,y) \right] ** \mathrm{h}(x,y)$$

$$= \mathrm{XY} \sum_{\mathrm{n}=-\infty}^{\infty} \sum_{\mathrm{m}=-\infty}^{\infty} \mathrm{g}(\mathrm{nX},\mathrm{mY}) \cdot \delta(x-\mathrm{nX}, y-\mathrm{mY})$$

$$** \frac{1}{\mathrm{XY}} \mathrm{sinc}\left(\frac{x}{\mathrm{X}}\right)\mathrm{sinc}\left(\frac{y}{\mathrm{Y}}\right)$$

Two Dimensional Sampling: Restoration of original function(2)

$$\hat{g}(x,y) ** h(x,y)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g(nX, mY) \cdot \mathrm{sinc}\left[\frac{1}{X}(x - nX)\right] \cdot \mathrm{sinc}\left[\frac{1}{Y}(y - mY)\right]$$

Each sample serves as a weighting for a 2D sinc function.

Nyquist/Shannon Theory:

We must sample at twice the highest frequency in x and in y to reconstruct the original signal.

(No frequency components in original signal can be $> \frac{1}{2X}$ or $> \frac{1}{2Y}$)

Two Dimensional Sampling: Example

80 mm Field of View (FOV)

256 pixels

Sampling interval = 80/256 = .3125 mm/pixel

Sampling rate = 1/sampling interval = 3.2 cycles/mm or pixels/mm

Unaliased for ± 1.6 cycles/mm or line pairs/mm

Example in spatial and frequency domain

Sampling process is Multiplication of infinite train of impulses III(x/Δx) with f(x)
or convolution of III(uΔx) with F(s)→ Replication of F(s)

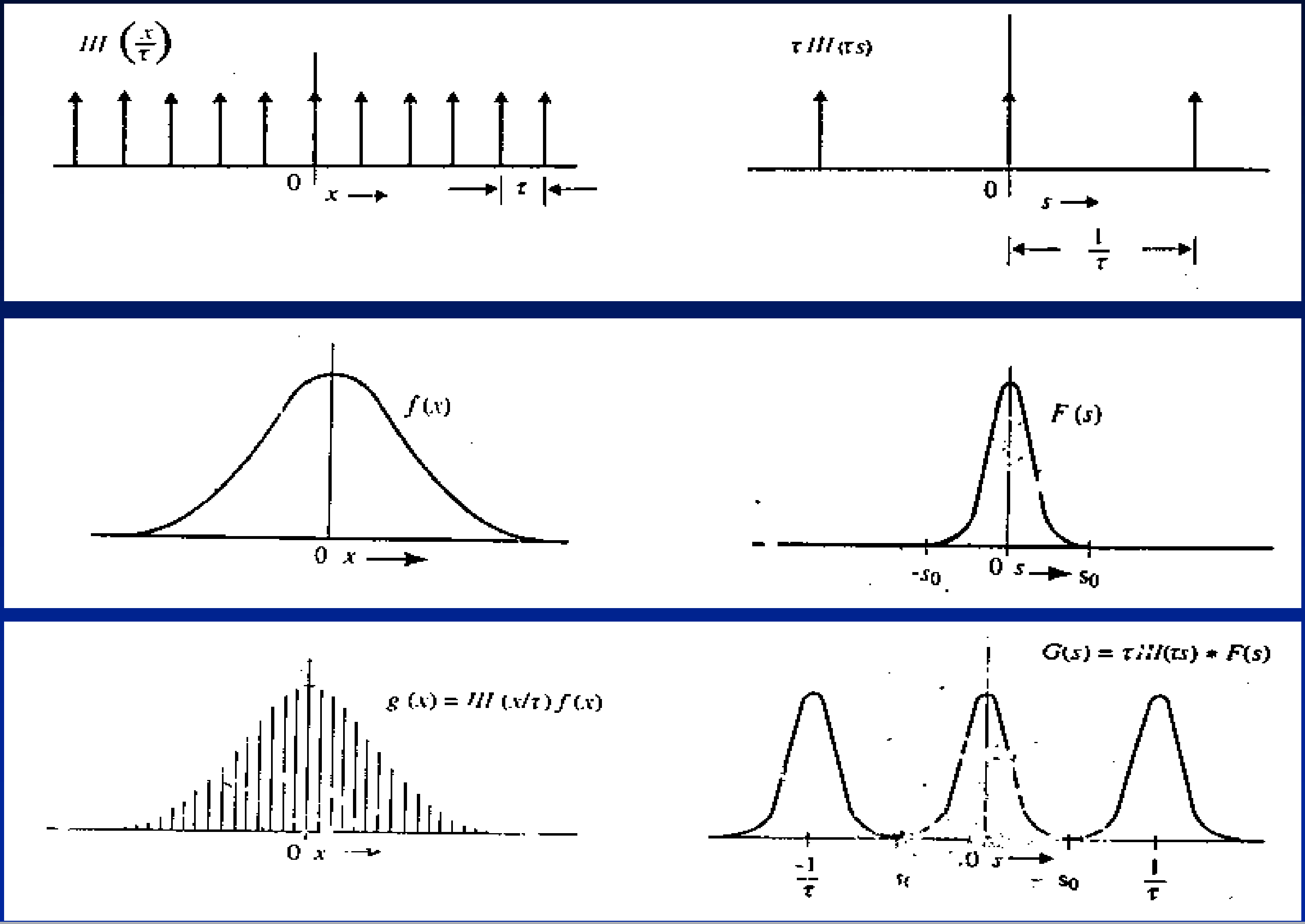
$$III\left(\frac{x}{\tau}\right)=\tau\sum\delta(x-n\tau)$$

In time domain

FT of Shah function By similarity theorem

$$FT\left[III\left(\frac{x}{t}\right)\right]=\tau III(\tau s)=\sum\delta\left(s-\frac{n}{\tau}\right)$$

Example in Time or Spatial domain

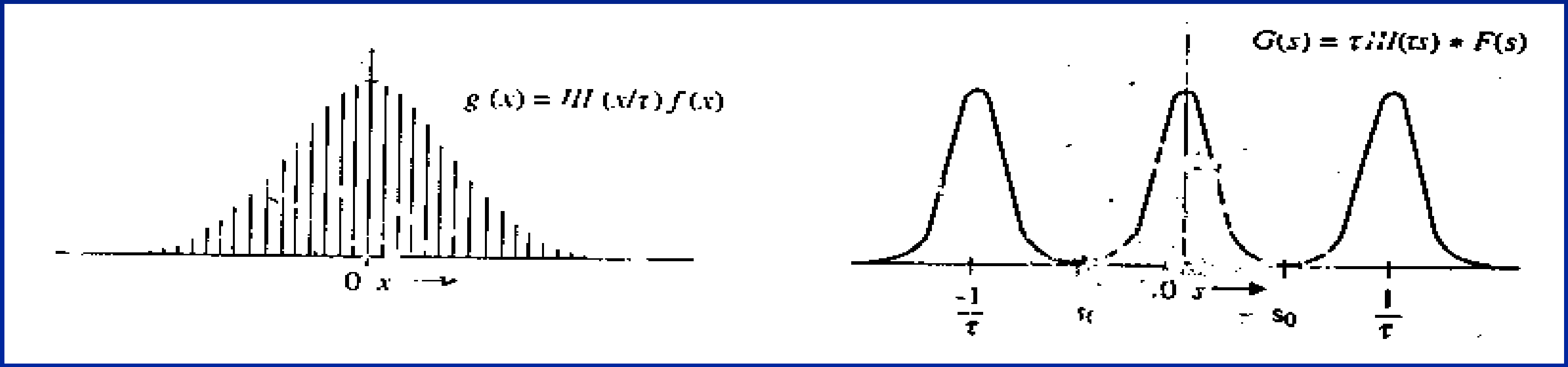


Sampling theorem

A function sampled at uniform spacing can be recovered if

$$\tau \le \frac{1}{2s_\circ}$$

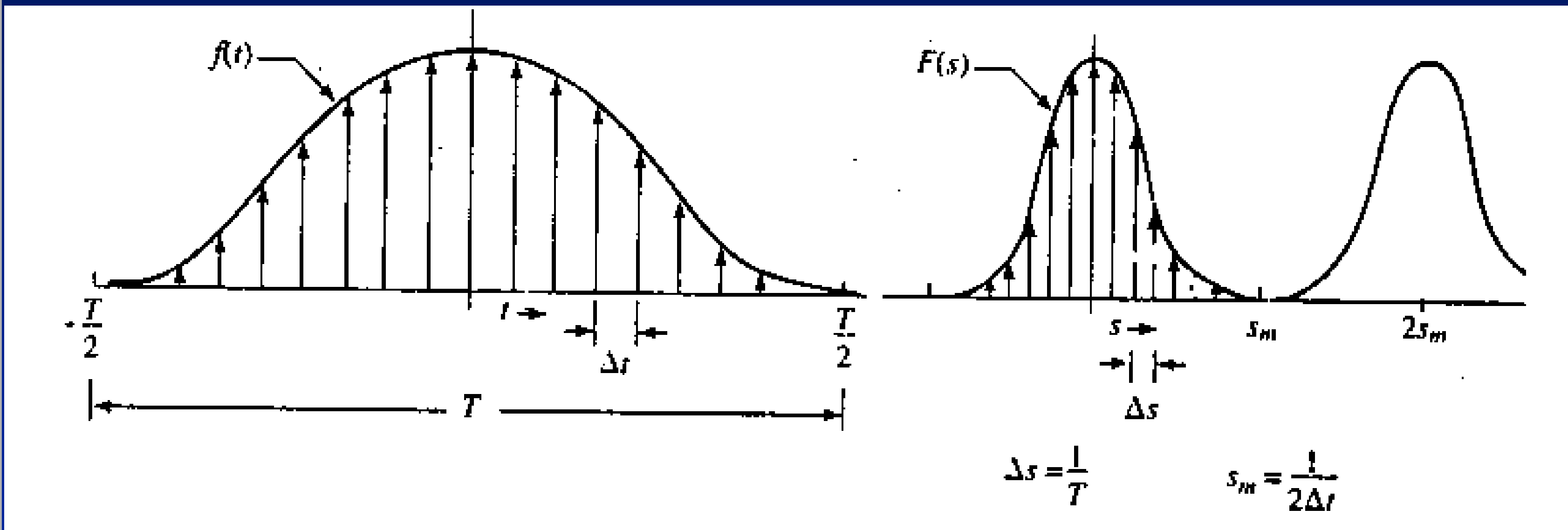
Aliasing: = overlap of replicated spectra



Properties of Sampling I

1) Truncation in Time Domain:

Truncation of f(x) to finite duration T =
Multiplying f(x) by Rect pulse of T =
Convolving the spectrum with infinite sinx/x



T = N Δt (Truncation window)
1/T = 1/NΔt= Δs spectrum sample spacing (in DFT)

Since Truncation is:

Multiply f(t) with window $\prod(\frac{t}{T})$

or convolve F(s) with narrow sin(x)/x
Therefore, it extends frequency range (to infinite)

→

Spectrum of truncation function is always infinite and Truncation destroy bond limitedness & produce alias.

This causes Unavoidable Aliasing

Properties of sampling II

2) There is a **Sampling Aperture** over which the signal is averaged at each sample point before applying Shah function

By convolve f(t) with aperture $$\frac{1}{\tau}\prod\left(\frac{t}{\tau}\right)$$

or multiply F(s) with $$\sin\frac{\pi s\tau}{\pi s\tau}$$

→ This reduces high frequency of signal

Properties of sampling III

3) Since Sampling is multiplication of shah function with continues function Or convolution of F(s) with

$$G(s) = \tau III(\tau s) * F(s)$$

- **Convolution of function with an impulse = copy of that function**
- **Replicate F(s) every $\frac{1}{\tau}$**

Properties of sampling IV

4) Interpolation or **Recovering original function (D/A)**

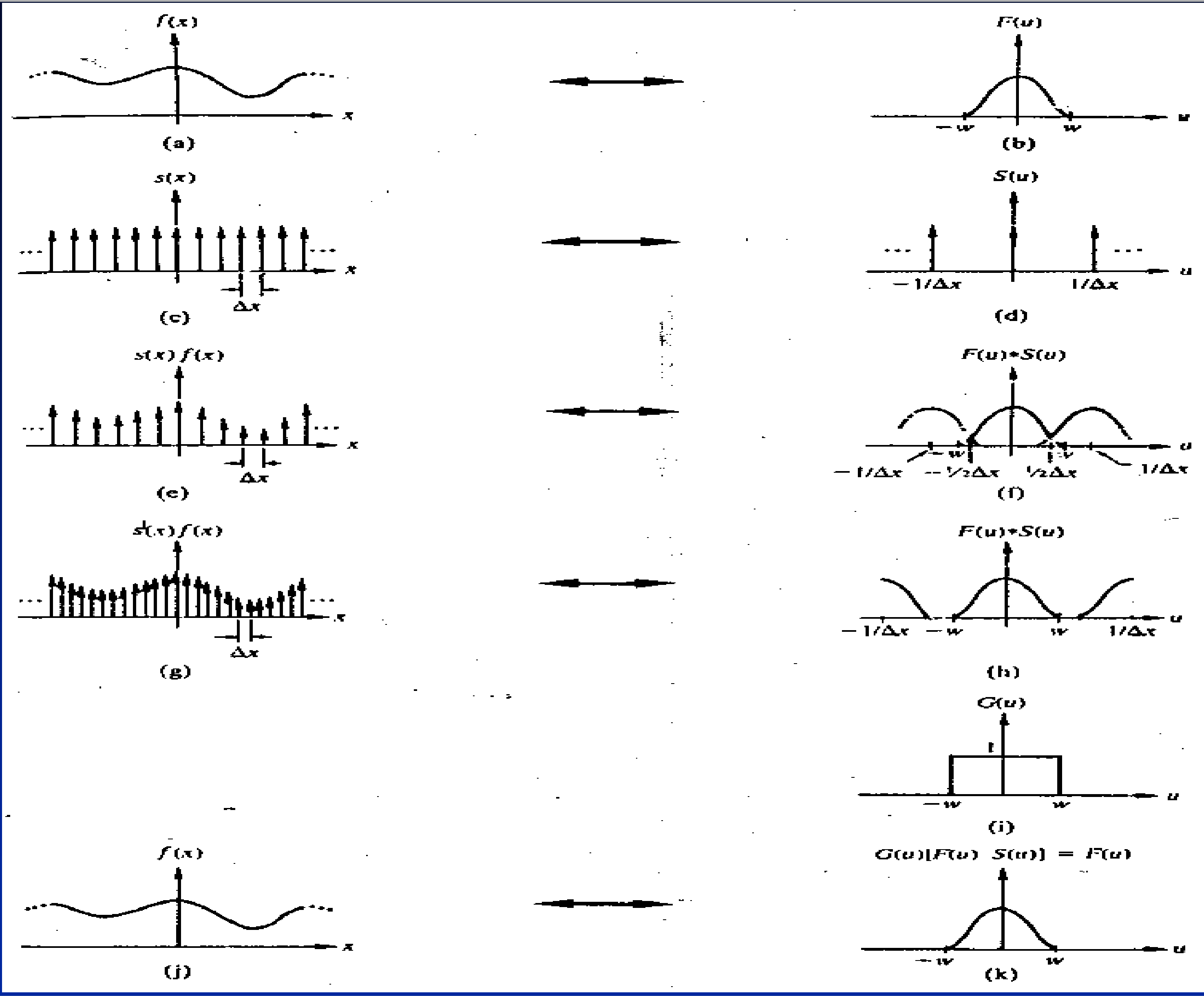
To recover original function, we should eliminate the replicas of F(s) and keep one.

Either Truncation in Freq should be done.

$$G(s)\prod\left(\frac{s}{2s_1}\right) = F(s) \qquad s_\circ \le s_1 \le \frac{1}{\tau} - s_\circ$$

Or convolving sampled g(x) with interpolation sinc

$$f(x) = FT^{-1}(Fs) = g(x) * 2s_1 \frac{\sin(2\pi s_1 x)}{2\pi s_1 x}$$



f(x)
x
(a)
F(u)
−w
w
u
(b)
s(x)
Δx
(c)
S(u)
−1/Δx
1/Δx
(d)
s(x) f(x)
Δx
(e)
F(u)*S(u)
−1/Δx
−½Δx
½Δx
1/Δx
(f)
s(x) f(x)
Δx
(g)
F(u)*S(u)
−1/Δx
−w
w
1/Δx
(h)
G(u)
1
−w
w
(i)
f(x)
(j)
G(u)[F(u) S(u)] = F(u)
−w
w
(k)

Review of Digitizing Parameters

Depend on digitizing equipment:

- **Truncation window → Max F.O.V of image**
- **Sampling aperture → Sensitivity of scanning spot**
- **Sampling spacing → Spot diameter (adjustable)**
- **Interpolation function → Displaying spot**

→

Review of Sampling Parameters

To have good spectra resolution (small Δs) and minimum aliasing, parameters N , T and Δt defined.

Δt as small as possible

T as long as possible → small Δs

→ compressed FT

To control aliasing:

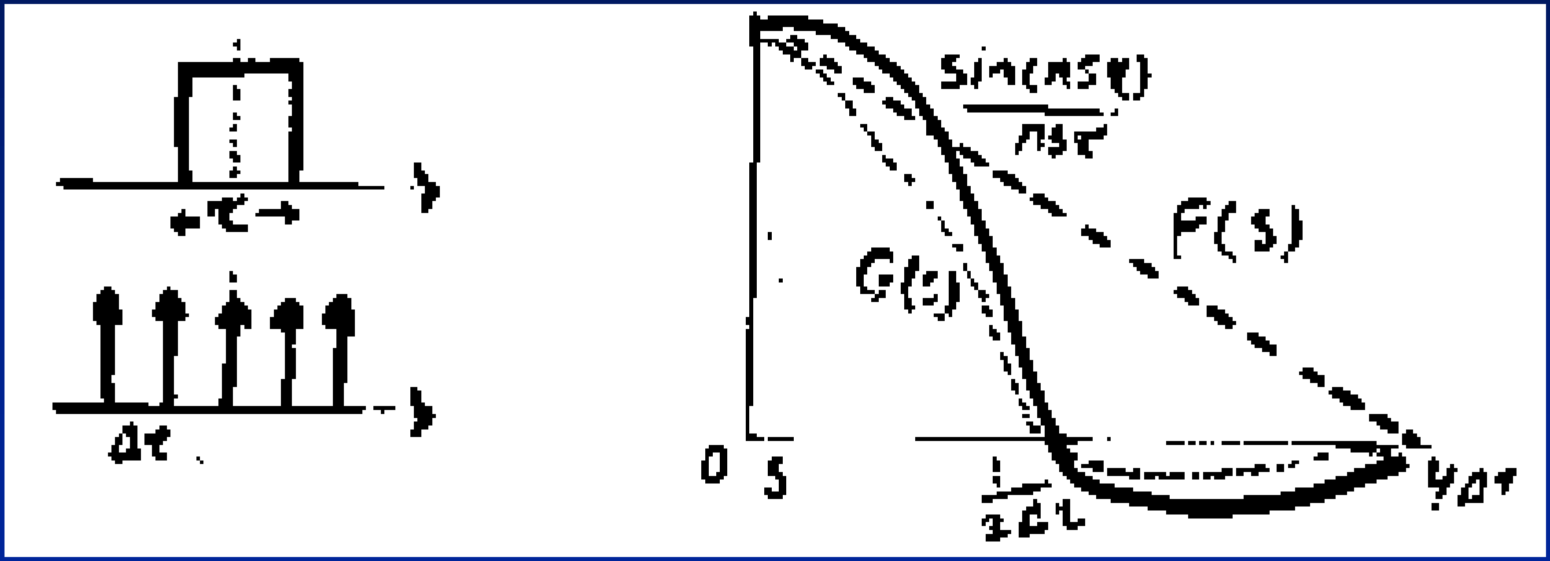
- Bigger sampling aperture
- Smaller sampling spacing (over same filter)
- Adjust image freq. S_m at most $S_m=1/2\Delta t$

Anti aliasing Filter:

1) Using rectangular aperture twice spacing

Energy at frequency above $S_0 > 1/2\Delta t$ is attenuated.

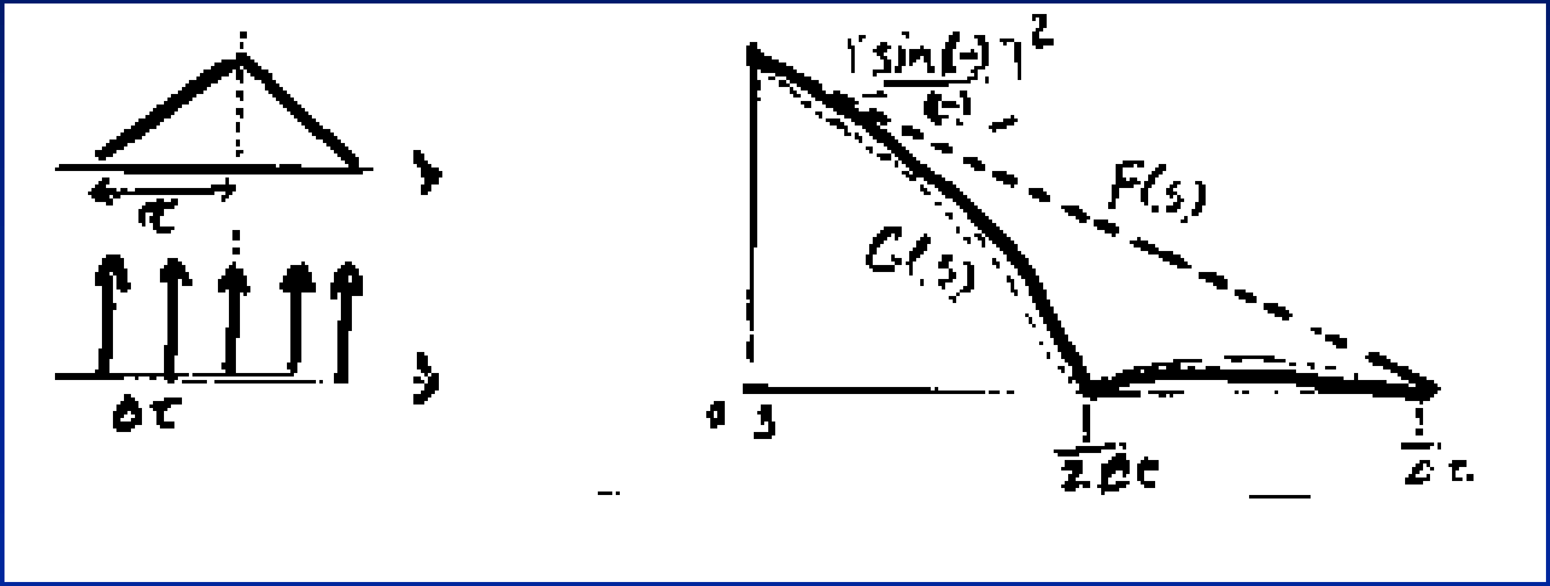
Original image freq. F(s) from $1/\Delta t$ reduce to $1/2\Delta t$



Anti aliasing Filter:

2) Using triangular aperture = 4 time of spacing

⟶ Dies of frequency above $1/2\Delta t$



Examples of whole Sampling Process on a Band limited Signal

Original signal:

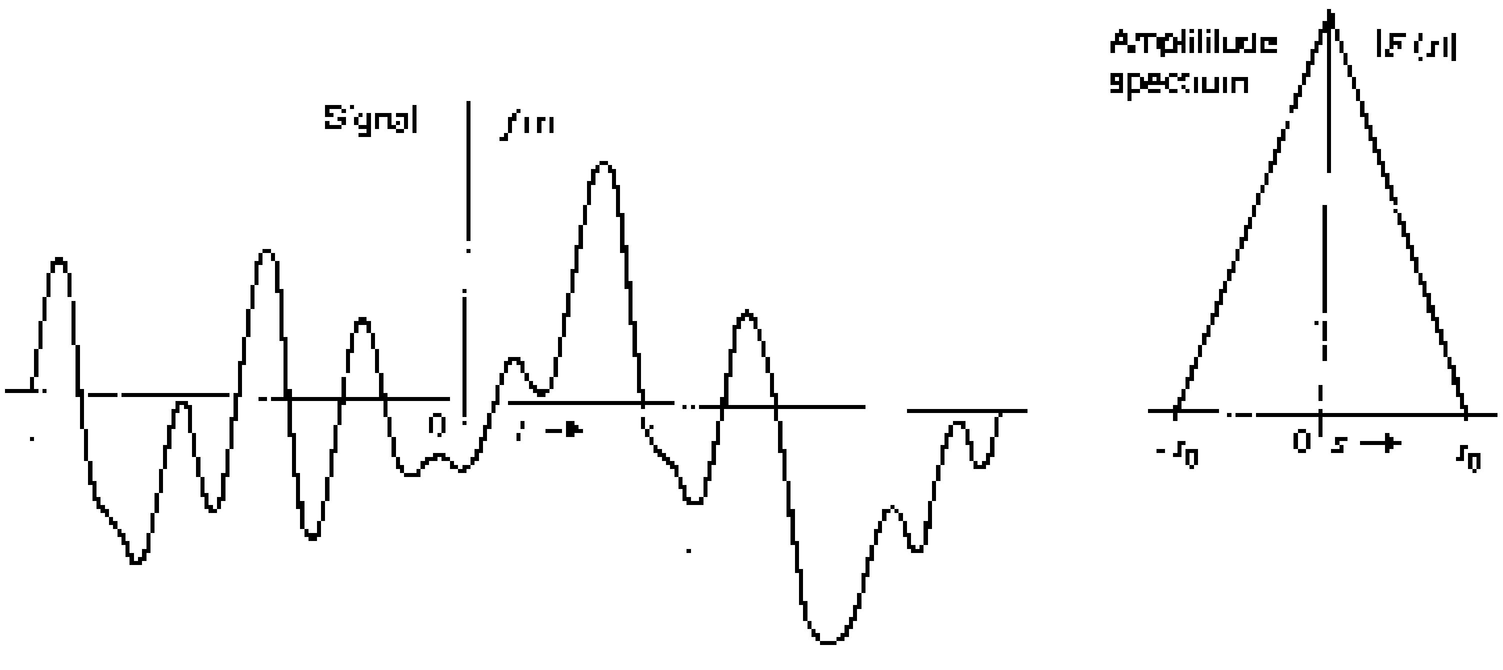
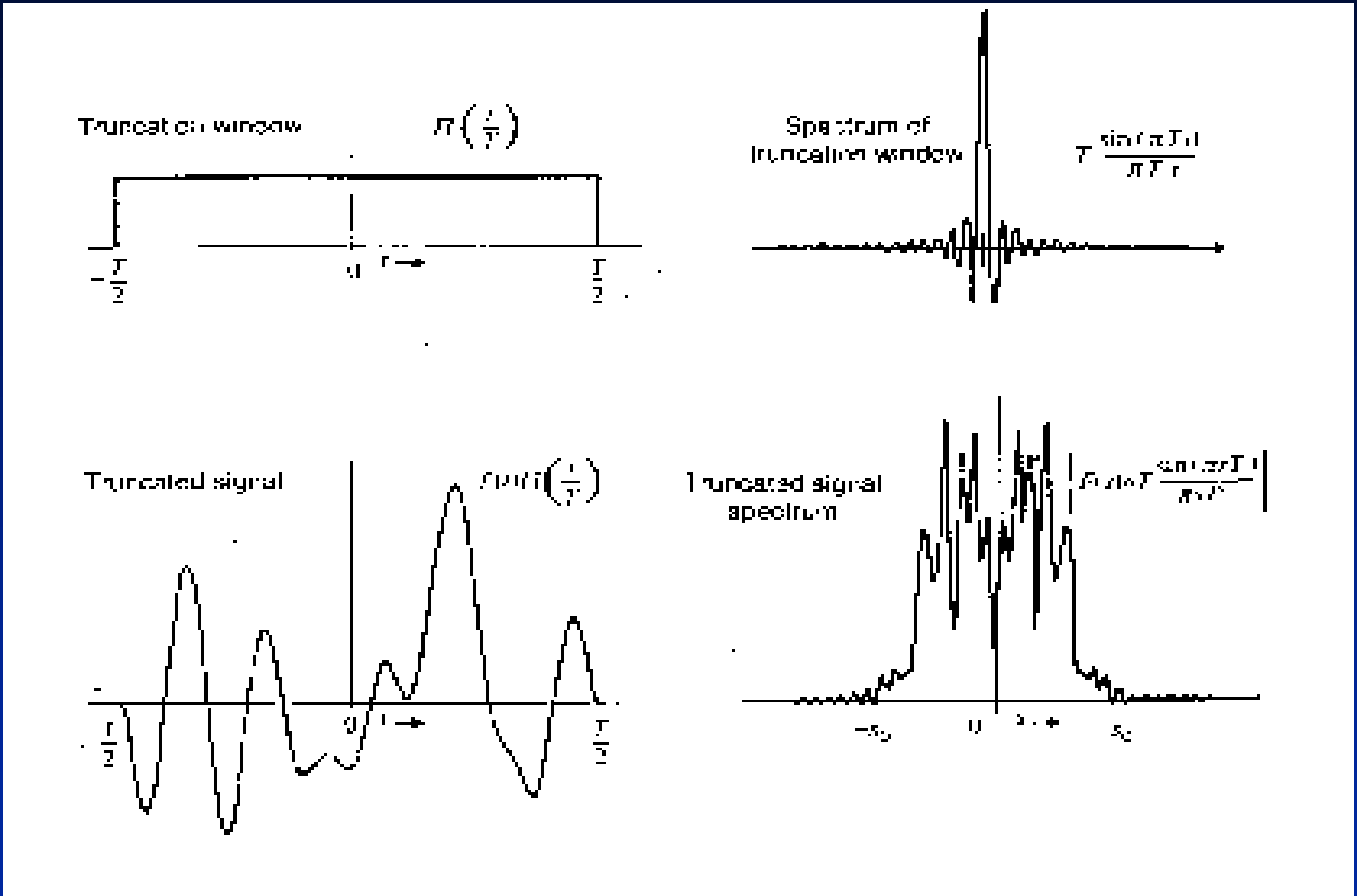
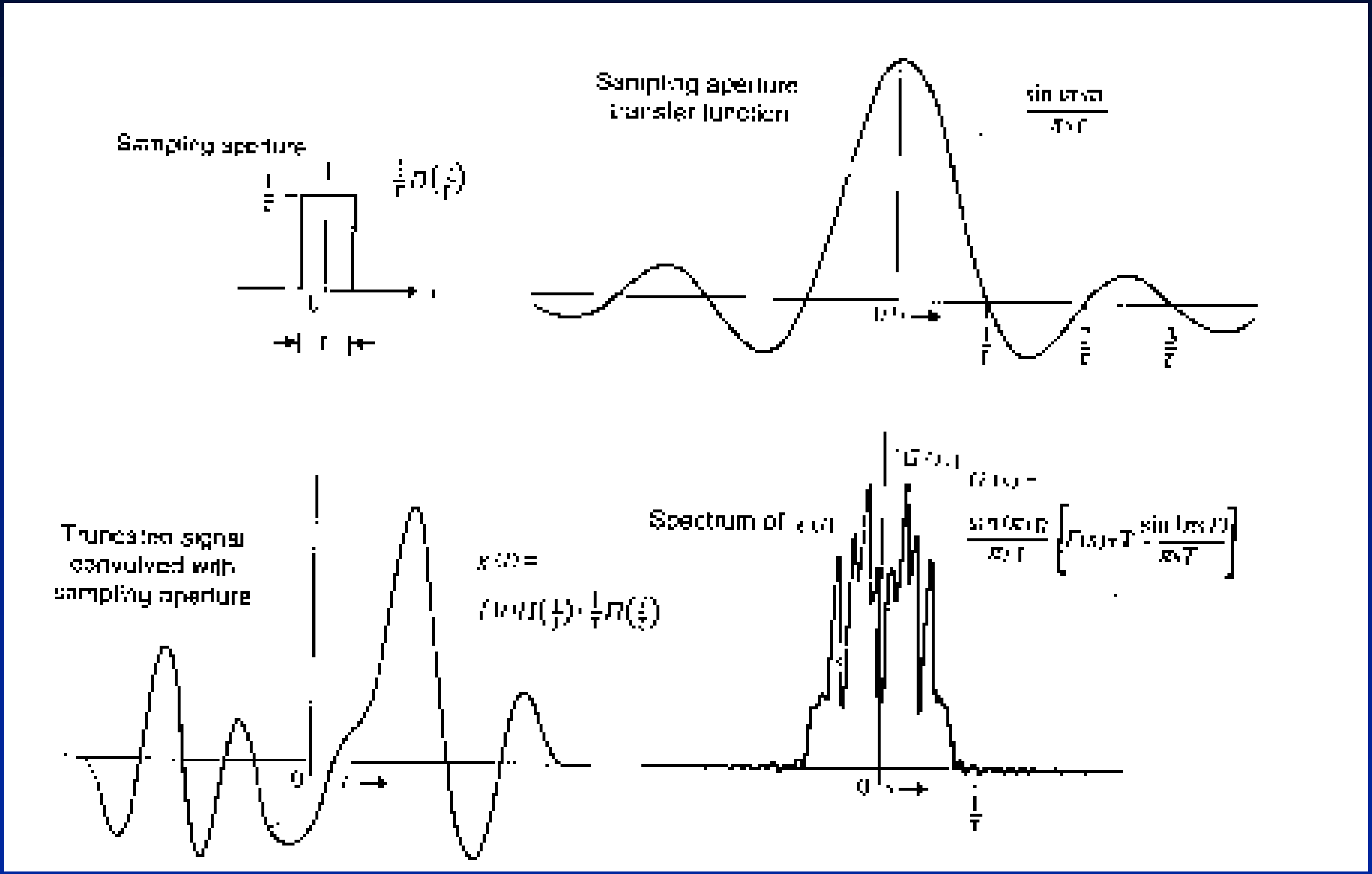


Figure 12–19 A signal and its spectrum

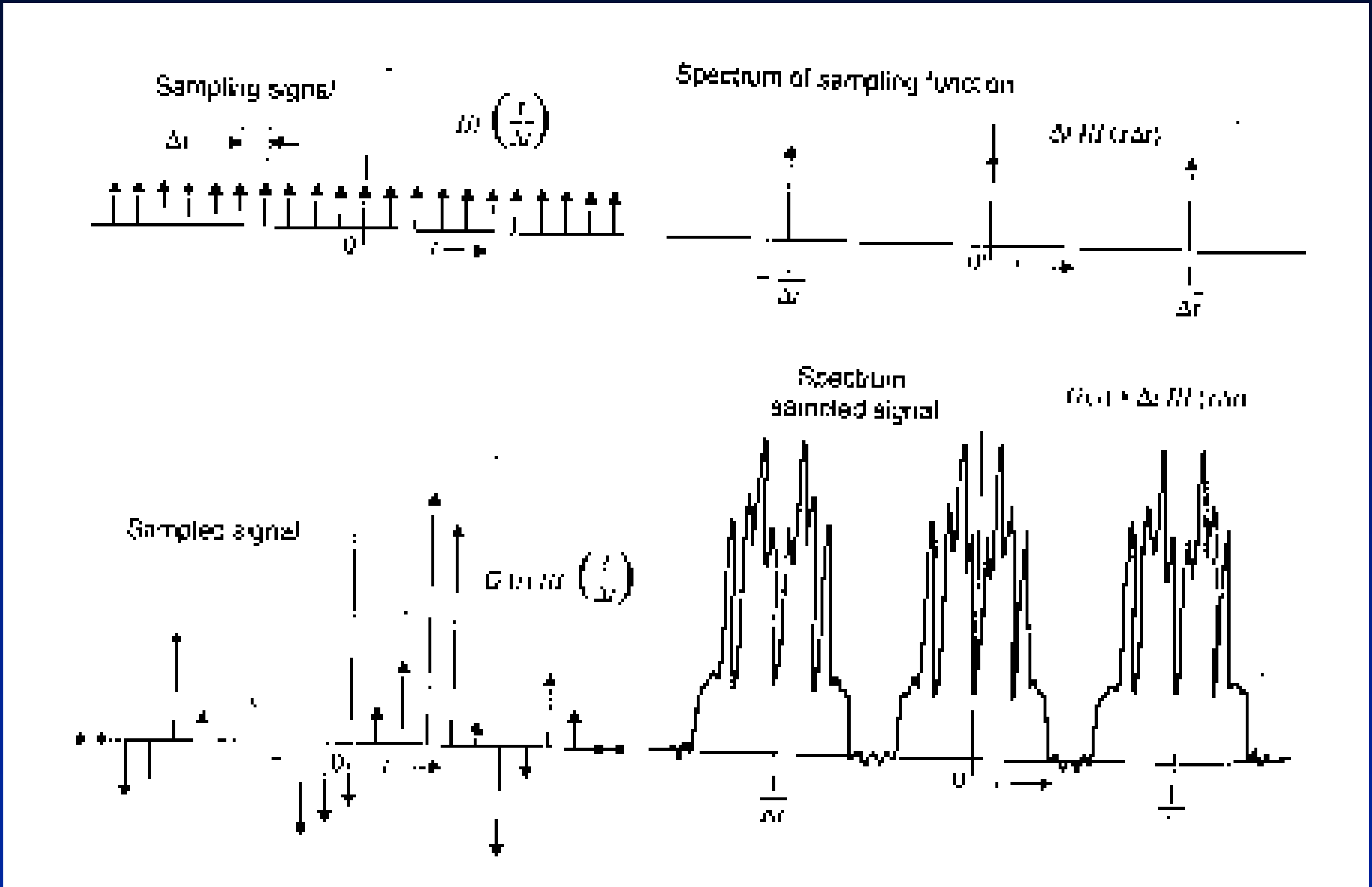
Truncating the signal:



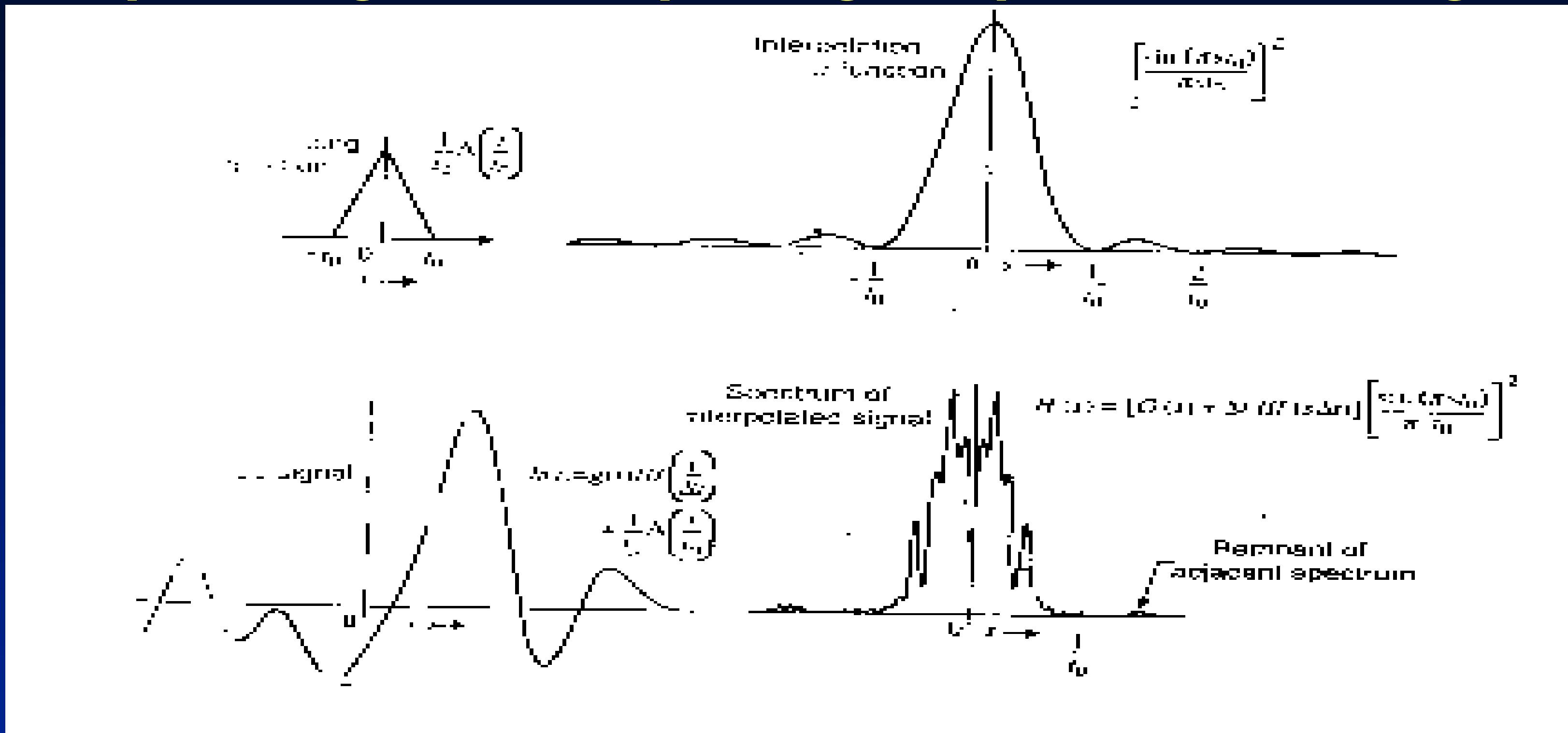
Convolving signal with sampling aperture:



Sampling the signal:



Interpolating the sample signal (to recover original)



$$h(t)=\left(\left\{\left[f(t)II\frac{1}{T}\right]*\frac{1}{\tau}II\left(\frac{1}{t}\right)\right\}III\left(\frac{1}{\Delta t}\right)\right)*\frac{1}{t_\circ}\Lambda\left(\frac{t}{t_\circ}\right)$$

$$I(s)=\left(\left\{\left[F(s)*T\frac{\sin(\pi sT)}{\pi sT}\right]\frac{\sin(\pi sT)}{\pi sT}\right\}*\Delta t III(s\Delta t)\right)\left[\frac{\sin(\pi s t_\circ)}{\pi s t_\circ}\right]^2$$

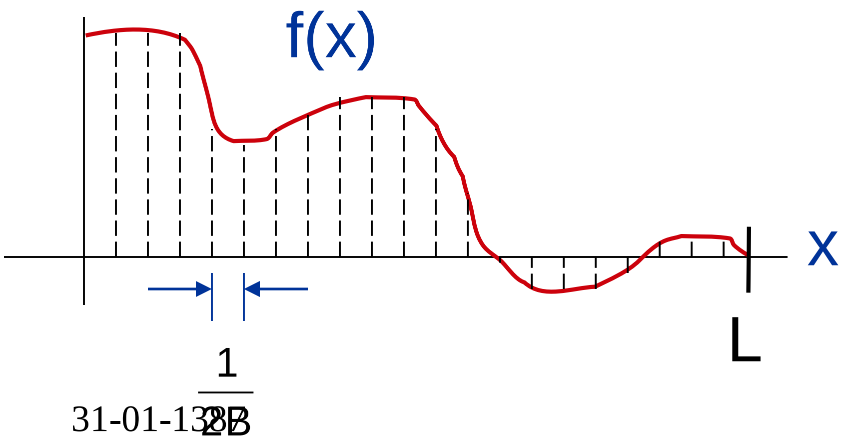
Discrete Fourier Transform

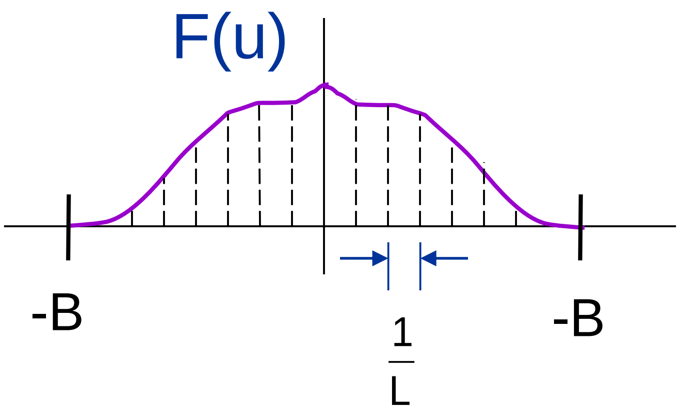
g(x) is a function of value for $-\infty < x < \infty$

We can only examine g(x) over a limited time frame, $0 < x < L$

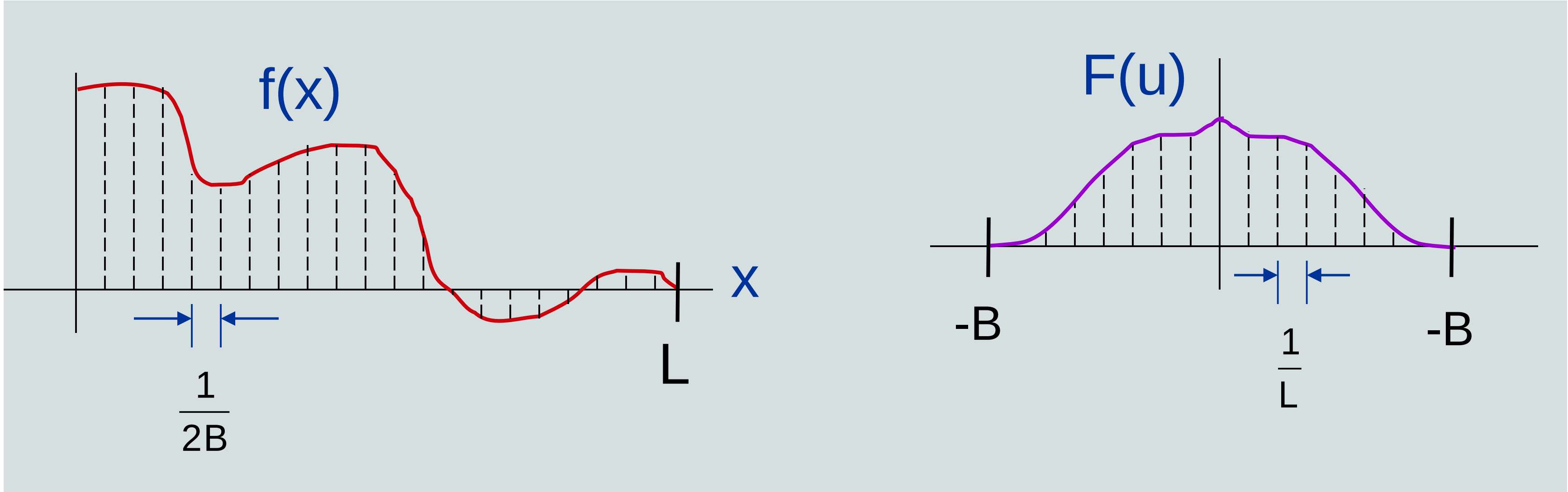
Assume the spectrum of g(x) is approximately bandlimited; no frequencies > B Hz.

Note: this is an approximation; a function can not be both time-limited and bandlimited.





Sampling and Frequency Resolution



We will sample at 2B samples/second to meet the Nyquist rate.

$$N = \frac{L}{1/2B} = 2BL$$

We sample N points.

$$\frac{\text{frequency range}}{\#\text{ of samples}} = \frac{2B}{N} = \frac{1}{L} = \text{frequency resolution}$$

Transform of the Sampled Function

$$\hat{f}(x) = \sum_{n=0}^{N-1} \frac{1}{2B} \cdot f\left(\frac{n}{2B}\right) \cdot \delta\left(x - \frac{n}{2B}\right)$$

$$\hat{F}(u) = \sum_{n=-\infty}^{\infty} F(u - 2nB)$$

Another expression for $\hat{F}(u)$ comes from $\mathcal{F}\left\{\hat{f}(x)\right\}$

$$\hat{F}(u) = \sum_{n=0}^{N-1} \frac{1}{2B} \cdot f\left(\frac{n}{2B}\right) \cdot \mathcal{F}\left\{\delta\left(x - \frac{n}{2B}\right)\right\}$$

Views input as linear combination of delta functions

$$\hat{F}(u) = \sum_{n=0}^{N-1} f(n)\ e^{-i \cdot 2\pi \cdot \frac{nu}{2B}}$$

where $f(n) \equiv \frac{1}{2B} f\left(\frac{n}{2B}\right)$

u is still continuous.

Transform of the Sampled Function (2)

$$\hat{F}(u) = \sum_{n=0}^{N-1} f(n) e^{-i \cdot 2\pi \cdot \frac{nu}{2B}}$$

where $f(n) \equiv \frac{1}{2B} f\left(\frac{n}{2B}\right)$

To be computationally feasible, we can calculate $\hat{F}(u)$ at only a finite set of points.

Since $f(x)$ is limited to interval $0<x<L$,
$\hat{F}(u)$ can be sampled every $\frac{1}{L}$ Hz.

Discrete Fourier Transform

$$\hat{F}\left(\frac{m}{L}\right) \equiv F(m) = \sum_{n=0}^{N-1} f(n) e^{-i2\pi \frac{nm}{2BL}}$$

$2BL = N =$ number of samples

Discrete Fourier Transform (DFT):

$$F(m) = \sum_{n=0}^{N-1} f(n) e^{-i2\pi \frac{nm}{2BL}} = \sum_{n=0}^{N-1} f(n) e^{-i2\pi \frac{nm}{N}}$$

Number of samples in x domain
= number of samples in frequency domain.

Periodicity of the Discrete Fourier Transform

DFT: $$F(m) = \sum_{n=0}^{N-1} f(n) e^{-i2\pi \frac{nm}{2BL}} = \sum_{n=0}^{N-1} f(n) e^{-i2\pi \frac{nm}{N}}$$

F(m) repeats periodically with period N

1) Sampling a continuous function in the frequency domain [F(u) ->f(n)]causes replication of f(n) (example coming in homework)

2) By convention, the DFT computes values for m= 0 to N-1

$m = 0$	DC component
$0 \text{ to } \frac{N}{2} - 1$	positive frequencies
$\frac{N}{2} + 1 \text{ to } N-1$	negative frequencies

Properties of the Discrete Fourier Transform

Let f(n) $\longrightarrow$ F(m)

1. Linearity If f(x) $\leftrightarrow$ F(u) and g(x) $\leftrightarrow$ G(u)

$$af(x) + bg(x) \rightarrow aF(u) + bG(u)$$

2. Shifting

$$D.F.T.\left\{f\ (n-k)\right\} \rightarrow \mathrm{F(m)}e^{-i\cdot 2\pi\cdot\frac{km}{N}}$$

Example : if k=1 $\longrightarrow$ there is a 2π shift as m varies from 0 to N-1

Inverse Discrete Fourier Transform

If f(n) $\longrightarrow$ F(m)

$$D.F.T.^{-1}\{\mathrm{F(m)}\} \equiv \tfrac{1}{\mathrm{N}} \sum_{m=0}^{\mathrm{N}-1} \mathrm{F(m)} \cdot e^{-i\cdot 2\pi \cdot \frac{nm}{\mathrm{N}}} = \mathrm{f}(n)$$

Why the 1/N ? Let's take a look at an example

f(n)= {1 0 0 0 0 0 0 0} N = 8 = number of samples.

$$\mathrm{F}(m) = \sum_{n=0}^{\mathrm{N}-1} \mathrm{f_n} \cdot e^{+i\cdot 2\pi \cdot \frac{nm}{8}}$$

$$= 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$

$= 1$ for all values of m

continued...

Inverse Discrete Fourier Transform (continued)

Now inverse,

$$f(n) = \tfrac{1}{N}\sum_{m=0}^{N-1} F(m) \cdot e^{+i\cdot 2\pi \cdot \frac{nm}{N}}$$

$$f(n) = \tfrac{1}{8}\sum_{m=0}^{N-1} F(m) \cdot e^{+i\cdot 2\pi \cdot \frac{nm}{8}}$$

$$f(0) = \tfrac{1}{8}\cdot 8 = 1$$

For n= 0,
kernel is simple

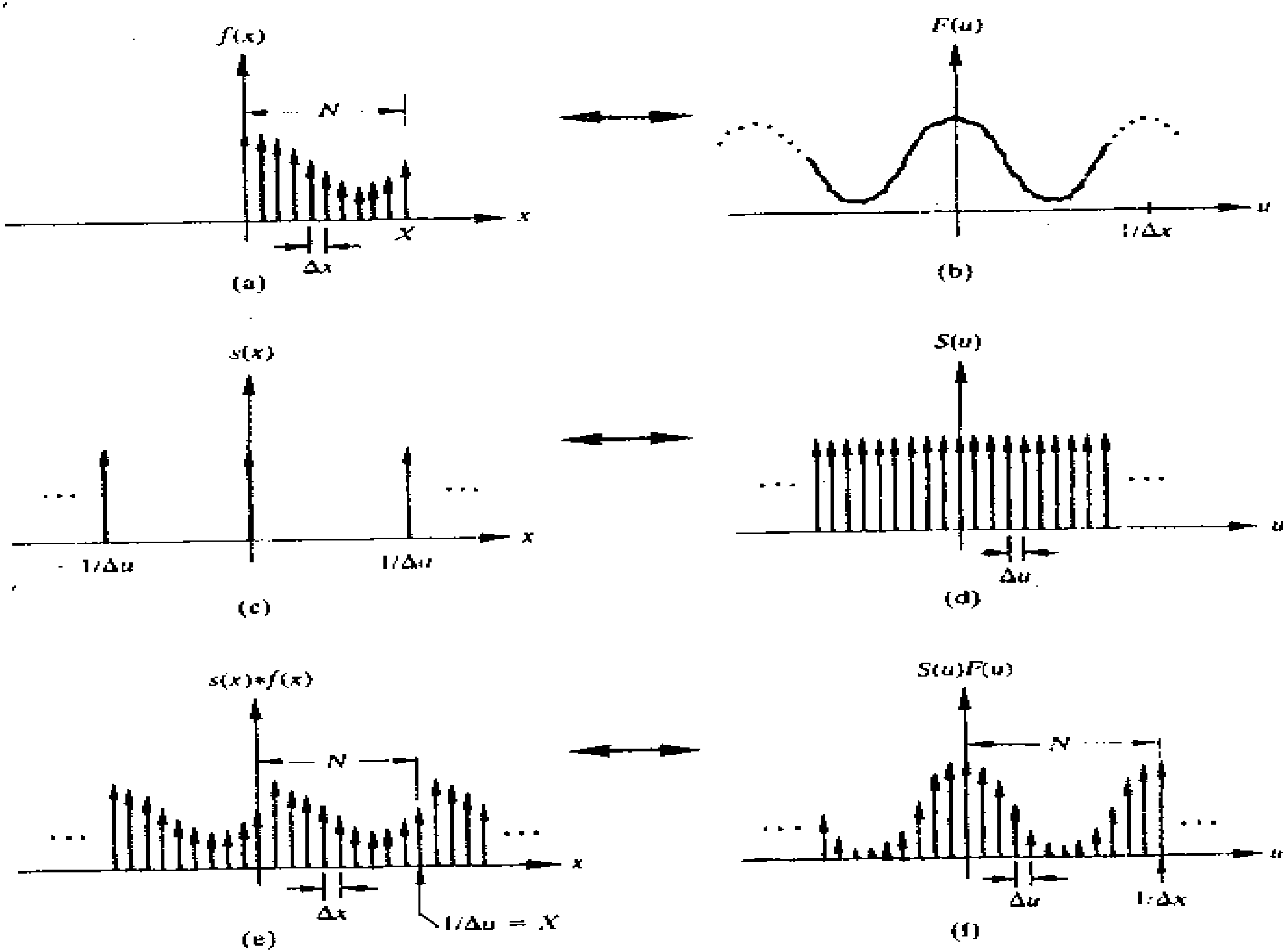
$$f(n) = \tfrac{1}{N}\sum_{n=0}^{N-1} F(m) \cdot e^{+i\cdot 2\pi \cdot \frac{nm}{N}}$$

For other values of n, this identity will help

$$\sum_{0}^{N-1} \frac{1}{r} = \frac{1-r^{N}}{1-r}$$

$$f(n) = \tfrac{1}{N}\left(\frac{1-e^{+i\cdot 2\pi \cdot \frac{mN}{N}}}{1-e^{+i\cdot 2\pi \cdot \frac{m}{N}}}\right)$$

$$f(n) = 0 \qquad \text{for m} \neq \ 0, N, 2N$$



f(x)
N
x
X
Δx
(a)
F(u)
u
1/Δx
(b)
s(x)
x
1/Δu
1/Δu
(c)
S(u)
u
Δu
(d)
s(x)*f(x)
N
x
Δx
1/Δu = X
(e)
S(u)F(u)
N
u
Δu
1/Δx
(f)